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Ollscoil na hÉireann Má Nuad

Quadratic and Non-Quadratic Stability Criteria for Switched Linear Systems

A dissertation

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by

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To Miles Joyce of Maamtrasna

und

Für meine Mutter und meinen Vater

Abstract

This thesis deals with the stability analysis of switched linear systems. Such systems are characterised by a mixture of continuous dynamics and logic-based switching between discrete modes. This system class appears in a large variety of control systems and has a wide field of applications in the modern industrial society.

While the application of switched control systems can be very beneficial, their stability analysis is often complex. Even though, switched system analysis has been studied extensively in the last decade, few analytical tools for stability have been developed. To date, the most common stability tools are based on numerical optimisation that provide little insight into the (in)stability properties of the process. The objective of this thesis is to develop stability tools that are readily applicable and provide some support for the design process of controllers for switched systems.

In this thesis stability criteria for hybrid systems that resemble many of the classical stability results for linear time-invariant systems are derived. The principal tool employed for stability analysis of the systems is Lyapunov theory: both quadratic and non-quadratic Lyapunov functions are used to derive compact eigenvalue conditions that guarantee stability of certain classes of switched systems. New results for second order switched systems are developed, a describing function technique for switched systems is presented, and pole-placement techniques for stabilising switched single-input single-output (SISO) PID control structures are derived. In addition, the results also lead to new more compact versions of well known SISO stability criteria for nonlinear systems of the Lur'e type. In this context, we show that well known criteria such as the Circle Criterion, Popov Criterion, and the KYP lemma can be evaluated in a compact manner.

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Chapter 1

Introduction and overview

In this introductory chapter, we motivate the study of switched and hybrid systems, and point out some of the issues associated with the stability of these systems. We also provide a rough overview of the nature of available stability results in the area, along with a brief outline of the work contained in the remainder of the thesis.

1.1 Introductory remarks and motivation

Over the past two decades dramatic progress in computing capabilities has resulted in the synthesis and implementation of increasingly complex dynamical systems. In this context, control systems often fulfill several objectives or interact in a network of decentralised control systems. Such systems typically exhibit simultaneously discrete and continuous dynamics and are known as hybrid dynamical systems. Control-problems that can be described in such a form can be found in applications in various industrial fields like aircraft control [BM00, BM01], traffic control [HV00, TPS98, GLS94], automotive control [Sho96, CGS04], and power systems [RBM+97, PL99]. It is fair to say that the theory of hybrid systems is by now a well-established research area with contributions from several research communities such as engineering, mathematics and computer science.

The subject of this thesis is a class of hybrid systems that is known as switched linear systems. The continuous dynamics of switched linear systems are described by a set of linear time-invariant differential equations which involve (at least partially) the same states. Each of these differential equations represent the dynamics of a linear time-invariant (LTI) system, often referred to as constituent systems of the switched linear system. The discrete dynamics are represented by some logic- or event-driven switching unit that alternates the linear dynamics at distinct time-instances. An example of a possible closed-loop controller structure is shown in Figure 1.1.

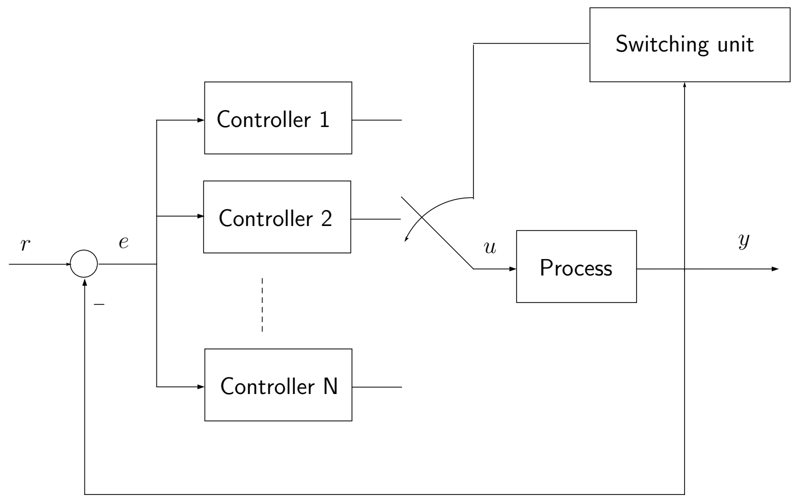


Figure 1.1: Example of the structure of a switched control system. Here a bank of N controllers is used to control the process. At any given time only one of the controllers is active in the closed-loop and supplies the control signal. The switching unit determine the controller to be active at any given time instant.

Switched control systems can be applied for a number of reasons. In the following list some scenarios are described that motivate the study of switched systems.

(i) **Switching process.** Maybe the most obvious reason for applying a switched control-scheme is given for the case where the process itself is inherently multi-modal such that its dynamics change (more or less) instantaneously over time. Examples for such behaviour can be found in mechanical systems where a gear-box is fundamental part of the dynamics. Consider the longitudinal dynamics of an automobile. In each gear the dynamics can be described by a different continuous model [Sho96]. A change of gear can then be viewed as a mode-switch of the overall longitudinal dynamics.

(ii) **Multiple control objectives.** Even when the plant itself does not exhibit switching dynamics, switched controllers can be used to meet multiple control objectives. This can be necessary when the process is exposed to changing environments or disturbances. A good example for an application with such changing objectives is the control of wind-turbine power generators [LLMS02, LSL+03]. For reasons of security and performance the generator operates in different modes depending on the current wind-speed.

(iii) **Performance and constraints.** Control-design is typically a trade-off between several objectives such as response speed, over-shoot, robustness, disturbance attenuation and control- and state-constraints. Some limitations of classical linear or non-linear controllers can be overcome by switching appropriately between individual controllers that are designed for specific tasks (for examples see [FGS97],[MK00] and references therein). For example, the performance of a PI controller can be improved by switching off the integral part of the controller when the error is large. For large errors the remaining proportional controller provides fast response, while the PI controller provides good low-frequency disturbance attenuation when the error is small.

(iv) **Adaptive control.** For processes with uncertainties and largely varying disturbances, adaptive controllers can be applied to account for the changing operating conditions. Switched dynamical systems arise naturally as a consequence of the introduction of the multiple-models, switching and tuning paradigm in [NB95, NB97]. Here, a number of different models are used to represent the various operating conditions. With each model a controller is associated whose parameters are tuned as long as the changes are small. When the conditions change rapidly, the model is selected that describes the current situation best and its associated controller is applied. Such a switching scheme allows for a much faster adaptation than classical single-model adaptive controllers.

While the application of switched controllers can undoubtedly be beneficial, the stability analysis of the resulting closed-loop systems is by no means trivial. At first sight, one might assume that the switched system only exhibits properties of the involved linear systems and therefore is stable whenever the constituent linear systems are stable. However, it is not hard to demonstrate that such a conjecture is false (see [LM99, DBPL00] for examples). In fact, the switching action between the constituent systems can induce a wide range of dynamical behaviour that cannot be observed in any of the constituent systems.

To assist intuition consider the following example of the "Car in the desert" borrowed from [SN98c] that illustrates a possible instability mechanism of the switched system.

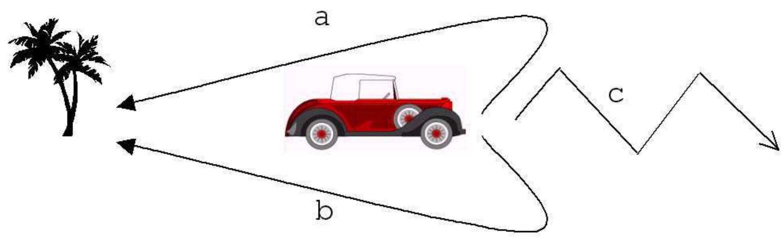


Figure 1.2: 'The car in the desert'

> *Imagine the situation depicted in Figure 1.2. A man is stranded with his car in the desert. As depicted he has two possible strategies (a and b) to reach the oasis and being saved from starvation. Both of those strategies can be considered as "stable" since they will lead to the rescue of the man. However, if the man initially follows strategy a but changes his mind after a while to follow strategy b, he might get further away from the oasis as indicated by path c. Continuation of changing his strategy in such a fashion would lead to catastrophic results. However, would he choose for example larger time-intervals between the change of his strategies (or simply never change his mind) he would reach the oasis safely.*

While the above example is somewhat frivolous, it still shows one of the many possible instability mechanisms of switched linear systems (known as chattering instability). The two strategies a and b represent trajectories in the state-space of the constituent stable LTI systems. A change of strategy corresponds to switching the dynamics from one vector-field to the other. Although the resulting trajectory c consists of fractions of trajectories belonging to strategy a or b, it exhibits a completely different qualitative behaviour. Furthermore, we observe that the same set of strategies (constituent systems) can result in stable and unstable trajectories, depending on the switching scheme applied. Therefore the switched system is only properly defined by the set of constituent systems in conjunction with admissible switching schemes.

This distinction motivates various approaches to the stability analysis of switched systems. Depending on the objectives and constraints of the application, different lines of inquiry are appropriate. Consider for example the case where we have little influence on the closed-loop dynamics of the constituent systems. In such situation it might be of interest to identify switching rules that result in stable behaviour. The opposite approach considers a class of admissible switching schemes and aims to design a set of controllers that ensure stability. Restriction of the class of admissible switching schemes usually results in a greater freedom for the design of the individual controllers.

This thesis is focussed on the stability analysis of switched linear systems with arbitrary switching schemes. Thus we derive conditions on the dynamics of the constituent systems such that the stability of the switched system is guaranteed. This is the most general case of the latter approach and is pertinent whenever the admissible switching schemes cannot be specified or a high degree of robustness (with respect to the switching scheme) is required. Moreover, as shall be discussed in a later section, the stability problem of switched systems with arbitrary switching is equivalent to robust stability of certain classes of parameter-varying systems. A pleasant 'by-product' of the analysis of switched systems for arbitrary switching is therefore, that the results are immediately applicable to the corresponding robustness problem.

1.2 Brief overview on existing stability approaches

The stability of the closed-loop system is probably the most fundamental objective for the control design. It is well known that efficient control-design methods can only be developed when the stability properties of the system are fully understood. In order to motivate the approaches in this thesis, this section provides a brief overview on the nature of available stability results for switched linear systems with arbitrary switching. A detailed review follows in Chapter 2.

The analysis of switched systems has received considerable attention from various research communities in the past decade. Most results are based on Lyapunov's stability theory which has played a dominant role in the analysis of dynamical systems for more than a century. Roughly speaking, Lyapunov theory guarantees stability of the dynamical system if a function (called Lyapunov function) with a certain set of properties can be associated with that system (see Section 2.4 for the full definitions). The crucial part of applying Lyapunov's theory is to find such a Lyapunov function or at least to establish that one exists. For linear time-invariant (LTI) systems this is rather straight forward since a simple eigenvalue-test reveals the existence of a quadratic Lyapunov function. Moreover, the existence of a quadratic Lyapunov function is necessary *and* sufficient for the asymptotic stability of an LTI system.

However, the analysis of switched linear systems is more complex. The introductory discussion indicates that the existence of a quadratic Lyapunov functions for each of the constituent LTI systems is not sufficient for the stability of the switched system. But it is well known that the switched system is stable if there exists some *common* Lyapunov function that satisfies the conditions of the Lyapunov theory simultaneously for all constituent systems. More recently, a number of converse theorems have been established [MP86, MP89, DM99], showing that such common Lyapunov function always exists when the switched linear system is stable for arbitrary switching. However, general conditions for determining the existence of a common Lyapunov function for switched systems are unknown. Therefore, most stability results consider some class of switched systems or a specific type of common Lyapunov function.

We can roughly categorise the available stability results for arbitrary switching into numerical and analytic conditions, based on either quadratic or non-quadratic Lyapunov functions.

The majority of results use common quadratic Lyapunov functions (CQLF) to establish stability. The most applicable analytic results are formulated in terms of eigenvalue properties of the system matrices (e.g. eigenvalue or matrix-pencil conditions) or apply to subsystems with specific properties such as commuting system matrices or triangular systems. While those conditions are very compact and simple to apply in practice, the greatest disadvantage lies in their restriction to a rather small class of systems. Numerical approaches for the existence of a CQLF are applicable to a much greater class of systems. This is mainly due to the fact, that the CQLF problem can be formulated as linear matrix inequalities (LMI) for which powerful solution-tools are available [BEFB94]. However, solving the respective LMIs only provides an answer to the existence (or non-existence) of a CQLF and does not provide any insights into the dynamical behaviour or guidelines for the controller design.

In contrast to LTI systems, the existence of a common Lyapunov function of the quadratic type is in general not necessary for the stability of switched systems. It is not hard to find stable switched systems for which no CQLF exists. In those cases, stability conditions that are based on the existence of a CQLF will lead to conservative results. This problem can be overcome by considering alternative types of Lyapunov functions as for example piecewise linear or piecewise quadratic Lyapunov functions. In the work on the converse Lyapunov theorems it is shown that such types of common Lyapunov functions always exist if the system is stable. The crucial question here is to determine the number of pieces needed to construct such Lyapunov function. In general, this problem appears to be rather difficult. Therefore, most results using these types of Lyapunov function concern switched systems where switching occurs along specified switching surfaces in the state-space. Such a structured state-space can be used as a guideline for the layout of the piecewise Lyapunov function. Analytic results using piecewise Lyapunov functions for arbitrary switching are very scarce. There are a few numerical approaches to finding a piecewise linear Lyapunov function. However, these algorithms also suffer from the problem that the number of partitions is unknown and effectively are not applicable to systems of order higher than three or four.

1.3 Overview and contributions

The subject of this thesis is the stability analysis of switched linear systems where arbitrary switching signals are admissible. As the previous discussion shows, there is a large number of stability results for this system class available in the literature. However, many of the conditions obtained are theoretical, non-constructive results or impose considerable restrictions on the control design for the systems. The main objective in this thesis is to contribute stability conditions that provide some insight into the stability (or instability) properties of the switched system and can support the design process of controllers for switched systems.

A second motivation for the work in this thesis stems from the fact that the majority of the stability tools available use quadratic forms to establish stability. This can lead to conservative results since the existence of a quadratic Lyapunov function is sufficient, but in general not necessary for stability. In other words, the considered system might well be stable even when a CQLF fails to exist. For this reason we choose alternative approaches in order to establish stability for a larger set of systems.

In the following Chapter 2 we formally define the switched systems considered in this thesis and introduce the relevant notions of stability. Furthermore, we discuss some typical stability problems that arise in the analysis of this system class and review the most relevant stability theory that is related to the work in this thesis.

In Chapter 3 a class of second-order switched systems is considered. We derive a simple stability condition (45° Criterion) which is based on eigenvalue properties of the constituent systems matrices. The formulation of the stability condition in terms of the eigenvalues has the advantage that the condition is invariant under co-ordinate transformations. We employ two types of Lyapunov functions to establish stability for this condition: quadratic Lyapunov functions and a type of piecewise linear Lyapunov functions. Clearly, the existence of either Lyapunov function is sufficient for the stability of the switched system. However, both Lyapunov functions are needed to derive the stability condition presented. The analysis reveals that the use of the piecewise linear Lyapunov function is particularly beneficial when the constituent LTI systems have real eigenvalues. For some of these cases, the quadratic Lyapunov function fails to exist. Moreover, the derivation of the results allows to directly construct such piecewise linear Lyapunov function for a given switched system. This could prove useful for the development of design-methods for controllers for switched systems.

While switched systems have only been explicitly studied for roughly a decade, their analysis is closely related to classes of nonlinear systems that have been the subject of earlier research. In Chapter 4 we consider a particular type of switched system that is related to the classical single-input single-output Lur'e system. We show that typical stability problems for these two system classes are equivalent such that stability results obtained for these systems are mutually applicable. A recent result in [SN03b] relates the Circle Criterion to a simple eigenvalue test of the product of the constituent system matrices. We extend this result to show that a large number of classical stability conditions for the Lur'e system can be formulated as eigenvalue conditions of a matrix product. In fact, any condition that relates the location of the Nyquist plot to a disk in the complex plane can also be expressed as an eigenvalue test (for example the sensitivity analysis of LTI systems). This can be useful for the robust design of controllers with several constraints.

The same system class is subject of the analysis in Chapter 5. Here we choose a frequency-domain approach which is motivated by a stability conjecture originally formulated in [PT74]. The authors use an approach similar to describing functions for stationary nonlinearities to detect the existence of periodic motion. In Chapter 5 we analyse this conjecture and examine a number of questions arising from it. In this context the existence of a periodic solution in the proximity of the stability boundary is of particular interest. While it remains an open question whether or not the non-existence of a periodic solution is sufficient for stability, we can establish that its absence is of some significance for the stability of the system. Based on this analysis we derive stability conditions that approximate the stability boundary arbitrarily close. Note, that these results are simultaneously valid for Lur'e systems and the corresponding class of switched systems.

In Chapter 6 we shift the focus from the pure analysis of the system to a design-method for a switched controller. We propose a controller architecture which is suitable to meet typical requirements for a switched process. We pursue two goals in this chapter. Firstly, we exploit the structure of the resulting closed-loop system to derive simplified stability conditions for systems with an arbitrary, finite number of subsystems. Secondly we derive conditions for the controller-design that minimise the transient responses when the plant switches under certain circumstances.

The work described in the thesis has led to a number of publications in international conference proceedings and peer-reviewed journals. In particular, results of Chapter 3 concerning the proposed piecewise linear Lyapunov function and the 45° Criterion for systems with real eigenvalues have been published in [WCS01], [WSC02b] and [WSC02a]. The extension of the matrix-product result in [SN03b] to multiplier criteria in the frequency domain in Chapter 4 has been published in [SCW03] and [SCW04]. The relation of the existence of periodic motion to the absolute stability problem in Chapter 5 has been discussed in [WFS03].

Chapter 2

Stability of switched linear systems

In this chapter we define the class of switched linear system and the notion of stability that are subject of the analysis in this thesis. Further, a number of problems regarding the stability of this system class are described, followed by a brief review of the known stability results on switched linear systems that are available in the literature.

2.1 Introductory remarks

Switched linear systems are defined by a set of linear time-invariant (LTI) systems and a switching mechanism that orchestrates between them. A key issue for the stability analysis of the switched system is the interaction of the piecewise constant dynamics and the switching mechanism. In fact, for some switching patterns this interaction can cause unstable behaviour in a switched system formed from a set of stable LTI systems and vice versa. For most of this thesis we are concerned with finding conditions on the family of subsystems that guarantee stable behaviour for arbitrary switching schemes.

In this chapter we formally define the class of switched systems considered in this thesis and the relevant notions of stability. Further we describe some major stability problems concerning switched systems that have been studied in the recent past. In

Section 2.4.2 a brief overview on Lyapunov theory in relation to switched systems is given and a number of fundamental converse theorems stated which form the theoretical basis of many stability results. In the last two sections we briefly review stability results available in the literature for switched systems with arbitrary and constraint switching signals, respectively.

2.2 Switched linear systems

The switched linear system consists of a finite set of linear time-invariant (LTI) systems

$$\Sigma_{A_i}: \quad \dot{x}(t) = A_i x(t), \qquad A_i \in \mathbb{R}^{n\times n}, \;\; i \in \mathcal{I} = \big\{1,\ldots,N\big\}. \tag{2.1}$$

We refer to Σ_{A_i} interchangeably as *constituent system* i, *subsystem* i or *mode* i of the switched system. The index-set $\mathcal{I}$ denotes the set of modes i. Since the subsystems are uniquely defined by their system matrix A_i, we will often refer to *the system* A_i, meaning the autonomous linear time-invariant system Σ_{A_i} defined by A_i.

Note, while each subsystem is described by an individual system matrix A_i, the state-vector $x \in \mathbb{R}^n$ is shared between them. At any given point in time one and only one subsystem Σ_{A_i} describes the evolution of the state $x(t)$. For most of this thesis we shall consider systems that switch arbitrarily between the constituent systems while the switching law may be unknown. However, we require that once the system switches into a given subsystem, it remains in that mode for some time interval. Although this time interval may be arbitrarily small, it excludes the possibility that an infinite number of switches occur in finite time, known as Zeno behaviour [Lib03].

The switching sequence describes the switching action between the constituent systems. The time instance where mode switches occur are given by a sequence of switching instances t_k. With every switching instant t_k we associate the mode $i_k \in \mathcal{I}$ that describes the subsystem that becomes active in the respective switching instant. Together they form the switching sequence given by the sequence of ordered pairs

$$(t_0, i_0), (t_1, i_1), \cdots, (t_k, i_k), \cdots$$

where $t_0 < t_1 < \ldots < t_k < \ldots$, and $i_k \in \mathcal{I}$, $i_k \neq i_{k+1}\; \forall\, k \geq 0$.

The k^{th} switching interval is given by $t_k \leq t < t_{k+1}$ where the switched system is in mode i_k. During that interval the state $x(t)$ evolves according to $\dot{x}(t) = A_{i_k} x(t)$.

Recall that the state vector x is common to the constituent systems and therefore is continuous at the switching instances. Thus the initial state $x(t_k)$ for $\dot{x}(t) = A_{i_k}x(t)$ is the terminal state of $\dot{x}(t) = A_{i_{k-1}}x(t)$,

$$x(t_k) = \lim_{t \to t_k, t < t_k} x(t).$$

The length of the k^{th} interval is denoted by $\tau_k = t_{k+1} - t_k > 0$.

The above elements fully describe the dynamics of the switched linear system. In order to describe the switched system in a closer form we introduce the switching signal which determines the interaction of the subsystems.

Definition 2.1 (Switching signal) *A switching signal $\sigma(t)$ is a piecewise constant function $\sigma : \mathbb{R}^+ \to \mathcal{I}$ with the following properties:*

(i) the points of discontinuity are the sequence of numbers $t_0 < t_1 < \ldots < t_k < \ldots$;

(ii) there exists a lower bound $\tau_{min} > 0$ for the interval between two consecutive discontinuities t_k, t_{k+1}, such that $t_{k+1} - t_k > \tau_{min}$ for all k;

(iii) $\sigma(\cdot)$ is continuous from the right, i.e. $\sigma(t) = i_k$ for $t_k \leq t < t_{k+1}$.

Using the elements described above we can define the switched linear system as a linear time-varying system with piecewise constant, linear dynamics given by a family of linear time-invariant systems.

Definition 2.2 (Switched linear system) *Let $\mathcal{A} = \{A_1, \ldots, A_N\} \subset \mathbb{R}^{n \times n}$ be a finite set of system matrices defining a family of linear time-invariant systems Σ_{A_i}, $i \in \mathcal{I} = \{1, \ldots, N\}$ and let $\mathcal{S}$ be the set of admissible piecewise constant switching signals $\sigma(\cdot) : \mathbb{R}^+ \to \mathcal{I}$. Then the switched linear system is defined by*

$$\Sigma_{\mathcal{A},\mathcal{S}} : \quad \dot{x}(t) = A(t)x(t), \tag{2.2}$$

where $A(t) \in \mathcal{A}$ and $A(t) \equiv A_{\sigma(t)}$, $\sigma(t) \in \mathcal{I}$ for all $t \in \mathbb{R}^+$ and all $\sigma(\cdot) \in \mathcal{S}$.

For most of this thesis we assume that the switching law allows for arbitrary switching, meaning that the set $\mathcal{S}$ contains all functions $\sigma(\cdot)$ with the properties in Definition 2.1. In these cases we shall omit the set $\mathcal{S}$ in the notation and simply refer to $\Sigma_{\mathcal{A}}$ as the switched system.

As a final piece of notation $x(t, t_0, x_0, \sigma)$ denotes the state vector x at the time t for a given initial condition $x_0 = x(t_0)$ and where the dynamics of the system switches according to σ. When the meaning is clear from the context we will simply use $x(t)$ to denote a trajectory or solution of the system. In the next section is shown that the solution $x(t, t_0, x_0, \sigma)$ of the switched system exists for all $t \in \mathbb{R}^+$ and any given initial state x_0, initial time t_0 and switching signal $\sigma(\cdot) \in \mathcal{S}$.

Representation of the solutions

At any time instant exactly one subsystem Σ_{A_i} is active and according to the assumptions, the solution $x(t)$ is continuous, albeit not continuously differentiable. Figure 2.1 shows an example for a state trajectory of a second-order switched system. Between two consecutive switching instances the trajectory has all characteristics of a linear time-invariant system. At the switching instances, the trajectory is possibly non-smooth but continuous.

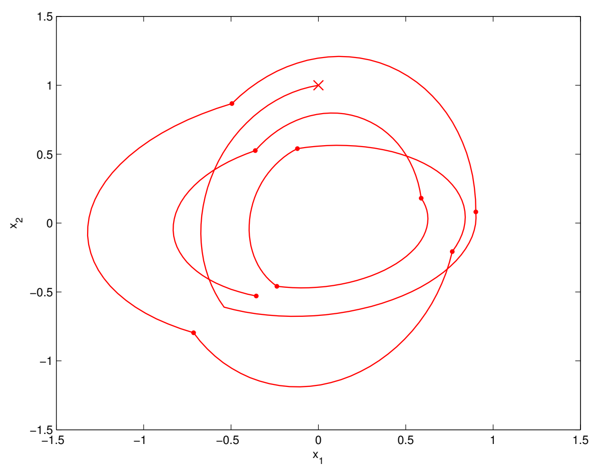


Figure 2.1: Sample-trajectory of an unforced second-order switched linear system. The initial state is $(0, 1)$ and the switching instances are indicated by a dot.

As for linear time-invariant systems we can write the solution of (2.2) for a given switching signal $\sigma(\cdot)$ and initial state x_0 as

$$x(t, t_0, x_0, \sigma) \quad = \quad \Phi_\sigma(t, t_0)\, x_0$$

where $\Phi_\sigma(t, t_0)$ denotes the transition matrix of the switched system (2.2) with switching signal $\sigma(\cdot)$.

The solution $x(t)$ can be constructed by piecing together the respective solutions of the constituent LTI systems. For the k^{th} interval $t_k \leq t < t_{k+1}$ we obtain

$$x(t) = e^{\left(A_{i_k}(t-t_k)\right)} x(t_k) .$$

Since $x(t_{k+1})$ is the initial state for the following interval we get for $t_{k+1} \leq t < t_{k+2}$

$$x(t) = e^{\left(A_{i_{k+1}}(t-t_{k+1})\right)} e^{\left(A_{i_k}\tau_k\right)} x(t_k)$$

with $\tau_k = t_{k+1} - t_k$.

The transition matrix $\Phi_\sigma(t, t_0)$ and with that the solution of (2.2) for any given initial state x_0 and any given switching signal $\sigma(\cdot)$, is uniquely defined by

$$\begin{aligned} x(t, t_0, x_0, \sigma) &= \Phi_\sigma(t, t_0)\, x_0 \\ \text{with} \qquad \Phi_\sigma(t, t_0) &= e^{\left(A_{i_k}(t-t_k)\right)} \prod_{l=0}^{k-1} e^{\left(A_{i_l}\tau_l\right)} \end{aligned}$$

where t_k is the last switching instant before t, i.e. $t_k \leq t < t_{k+1}$.

Non-autonomous systems

Thus far we defined the switched linear system as an unforced system without input and output, commonly referred to as an autonomous system. Of course, for control systems the input-output behaviour is of major interest. We therefore introduce the non-autonomous switched system.

Similar to (2.1) we consider the set of constituent linear time-invariant input-output systems given by

$$\dot{x}(t) = A_i x(t) + B_i u(t) \tag{2.3a}$$

$$y(t) = C_i x(t), \qquad \forall\ i \in \mathcal{I} \tag{2.3b}$$

where $u(t) \in \mathbb{R}^r$ denotes the input and $y(t) \in \mathbb{R}^p$ is the output of the system. The matrices B_i, C_i are of according dimensions. In the special case that $r = p = 1$ we speak of (2.3) as a single-input single-output (SISO) system.

The non-autonomous switched system is then defined by the family of time-invariant input-output systems (2.3) for $i \in \mathcal{I}$. Thus, the input-output behaviour in each mode $i \in \mathcal{I}$ is defined by the matrix triple (A_i, B_i, C_i). The input $u(t)$ and the output $y(t)$ signals are shared by all constituent systems.

Including the switching signal $\sigma(\cdot) \in \mathcal{S}$ we obtain for the non-autonomous switched system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2.4a}$$

$$y(t) = C(t)x(t) \tag{2.4b}$$

where $A(t) \equiv A_{\sigma(t)}$, $B(t) \equiv B_{\sigma(t)}$ and $C(t) \equiv C_{\sigma(t)}$ for all $t \in \mathbb{R}^+$ and $\sigma(t) \in \mathcal{I}$. Note, that the matrices $A(t)$, $B(t)$, $C(t)$ switch synchronously at exactly the same time instances t_k. Thus, the non-autonomous switched system is defined by the set of input-output systems $\big\{(A_i, B_i, C_i)\big\}$, $i \in \mathcal{I}$ and the set of admissible switching signals $\sigma(\cdot) \in \mathcal{S}$.

2.3 Stability theory of switched systems

Stability is a fundamental requirement for all control systems. As we shall see, there are a number of questions pertaining to the stability of switched linear systems that are as yet unanswered. Before we begin our discussion of these question we shall now introduce the formal definitions of the types of stability that are considered in this thesis.

2.3.1 Internal stability

Internal stability considers stability of the autonomous system in view of an equilibrium point x_e. A point $x_e \in \mathbb{R}^n$ is called equilibrium of the autonomous system (2.2) if $A_i x_e = 0$ for all $i \in \mathcal{I}$. In particular, the origin is always an equilibrium point of (2.2).

Definition 2.3 (Uniformly stable equilibrium) *The origin is said to be a uniformly stable equilibrium of* (2.2) *if for any given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\|x_0\| < \delta(\varepsilon) \quad \Rightarrow \quad \|x(t, t_0, x_0, \sigma)\| < \varepsilon \qquad \forall\, t \geq t_0\,,\ \forall\, \sigma(\cdot) \in \mathcal{S}.$$

Note that $\delta(\varepsilon)$ does neither depend on t_0 nor on $\sigma(\cdot)$. Hence, stability of the equilibrium is uniform with respect to the initial time and the switching signal $\sigma(\cdot)$.

If there exists a region around the origin such that all trajectories starting within that region converge to the origin, we speak of an *attractive* origin.

Definition 2.4 (Uniformly attractive equilibrium) *The origin is said to be a uniformly attractive equilibrium of* (2.2) *if there exists a* $\delta_a > 0$ *such that*

$$\|x_0\| < \delta_a \quad \Rightarrow \quad x(t, t_0, x_0, \sigma) \to 0 \quad as \quad t \to \infty \qquad \forall\, t_0 \geq 0\,,\ \forall\, \sigma(\cdot) \in \mathcal{S}.$$

Again, uniformity applies with respect to the initial time and the switching signal.

Combining both notions of stability we obtain asymptotic stability.

Definition 2.5 (Uniform asymptotic stability) *The origin is said to be a uniformly asymptotically stable equilibrium if it is uniformly stable and uniformly attractive.*

Figure 2.2 shows an illustration of the above notions of stability.

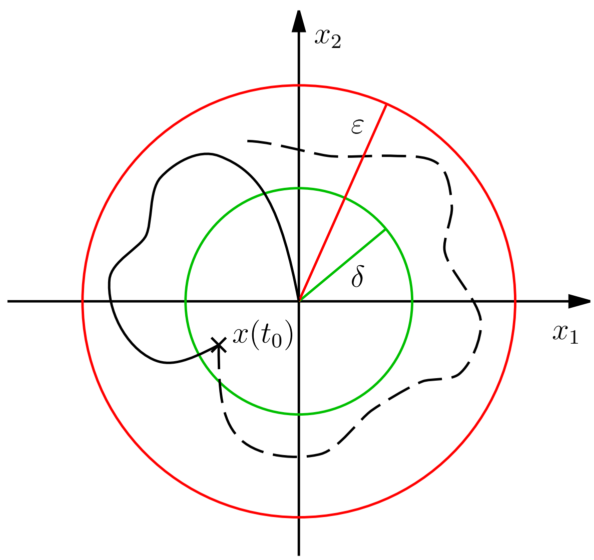


Figure 2.2: Illustration of stability definitions. The dashed line stays within the ball ε, but will not converge to the origin (equilibrium). Systems with such trajectories are called stable. The solid trajectory stays within the ε-ball and converges to the equilibrium, i.e. such systems are asymptotically stable.

An even stronger property is *exponential stability* which refers to the rate of convergence.

Definition 2.6 (Uniform exponential stability) *The origin is said to be a uniformly exponentially stable equilibrium if there exist real numbers* $a, b > 0$ *such that all solutions of* (2.2) *satisfy*

$$\|x(t, t_0, x_0)\| \leq a\|x_0\|e^{-bt} \qquad \forall\, t \geq t_0\,, \forall\, \sigma(\cdot) \in \mathcal{S}.$$

Some remarks are in order for the case where $\mathcal{S}$ contains all switching signals as in Definition 2.1:

(i) For linear time-varying systems asymptotic stability is equivalent to *global* asymptotic stability, i.e. the region of attraction is the whole state-space [Kha96].

(ii) For switched linear systems uniform attractivity, uniform asymptotic stability and uniform exponential stability are all equivalent [Ang99, DM99].

Note, that in all definitions uniformity is with respect to all admissible switching signals $\sigma(\cdot) \in \mathcal{S}$. For most of this thesis we are interested in stability of arbitrary switching between the constituent systems, i.e. $\mathcal{S}$ contains all signals as defined in Definition 2.1. However, when a system fails to have an asymptotically stable equilibrium in $\mathcal{S}$, we might still be able to find a subset $\mathcal{S}' \subset \mathcal{S}$ of switching signals for which stability can be established.

For the remainder of this thesis we omit the term "uniform" and use asymptotic stability and exponential stability interchangeably, meaning uniform exponential stability.

2.3.2 Input-Output stability

All the above notions of stability consider unforced systems and describe the behaviour of the state trajectories $x(t)$ for initial states $x_0 \neq 0$. In contrast to that, input-output stability addresses the stability of the non-autonomous system with initial state $x_0 = 0$ and relates the output behaviour of the system to bounded inputs. If all bounded inputs $u(\cdot)$ result in outputs $y(\cdot)$ that are themselves bounded, we speak of bounded-input bounded-output stability (BIBO-stability).

Definition 2.7 (BIBO-stability) *The non-autonomous switched linear system* (2.4) *is BIBO-stable if there exists a finite constant $\gamma > 0$ such that for any initial time t_0, any switching signal $\sigma(\cdot) \in \mathcal{S}$ and any bounded input signal $u(\cdot)$, the corresponding output signal $y(\cdot)$ satisfies*

$$\sup_{t \geq t_0} \|y(t)\| \leq \gamma \sup_{t \geq t_0} \|u(t)\|$$

Essentially, BIBO-stability provides that input signal cannot be amplified by a gain greater than some finite constant γ.

There are strong relations between input-output stability and internal stability (see e.g. [AM69] for a thorough discussion). Most relevant in our context is, that the non-autonomous system (2.4) is bounded-input bounded-output stable, if the corresponding autonomous system (2.2) is uniformly asymptotically stable [Rug96]. Therefore, we will focus our analysis on the internal stability of the autonomous switch system.

2.3.3 Stability problems for switched linear systems

Determining asymptotic stability for a single LTI system is rather simple. It is well known that the system

$$\dot{x}(t) \quad = \quad Ax(t), \qquad A \in \mathbb{R}^{n \times n}$$

is asymptotically stable if and only if the eigenvalues of A lie in the open left half of the complex plane [Kai80]. Matrices A with such properties are referred to as Hurwitz matrices. However, for switched systems with arbitrary switching it is not sufficient for stability that all constituent systems Σ_{A_i} are stable LTI systems. Instead, the switching between the modes can cause instability even if the LTI systems on their own are stable. This is illustrated by the following example.

Example 2.1 *Consider the set of system matrices* $\mathcal{A} = \{A_1, A_2\}$ *with*

$$A_1 = \begin{pmatrix} -0.5 & -1.0 \\ 15.0 & -1.5 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

It is readily verified that both matrices are Hurwitz and therefore the respective LTI systems are asymptotically stable.

We choose a periodic switching signal where the switched system is first in mode 1 for $\frac{2\pi}{25}$ *time-units and then in mode 2 for* $\frac{8\pi}{25}$ *time-units. We apply the following switching sequence*

$$\Big(0, 1\Big), \Big(\frac{2\pi}{25}, 2\Big), \Big(\frac{10\pi}{25}, 1\Big), \Big(\frac{12\pi}{25}, 2\Big) \dots \tag{2.5}$$

The trajectory of the switched system for this switching signal and the initial state $(1, 1)^\mathsf{T}$ *is shown as a dashed line in Figure 2.3. Clearly, this switching signal results in an unstable trajectory.*

However, choosing a slightly different switching signal can result in opposite stability behaviour. Choose the periodic switching sequence

$$\Big(0, 1\Big), \Big(\frac{4\pi}{25}, 2\Big), \Big(\frac{10\pi}{25}, 1\Big), \Big(\frac{14\pi}{25}, 2\Big) \dots \tag{2.6}$$

This switching signal results in a trajectory that converges to the origin, shown as a solid line in Figure 2.3. Note that the only difference between the switching sequences (2.5) *and* (2.6) *is the ratio of the time that the system stays in each mode.* □

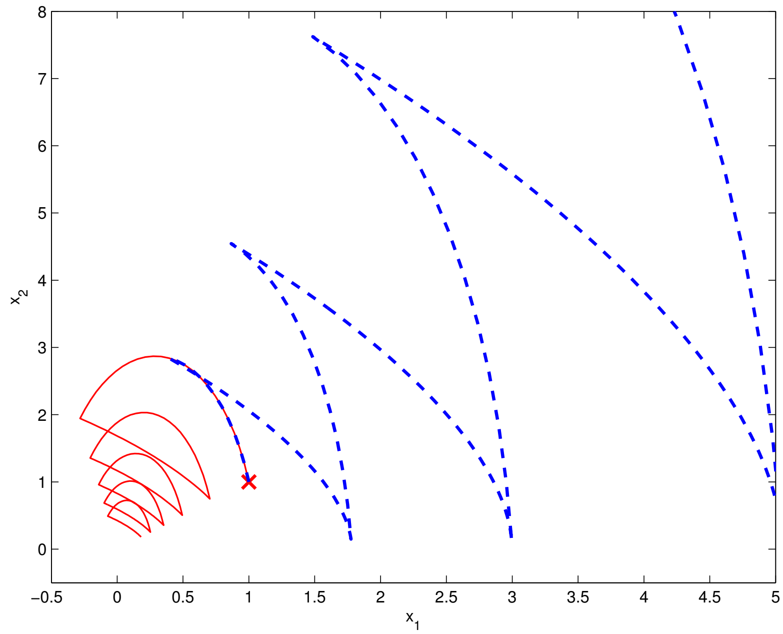


Figure 2.3: Sample trajectories of the switched system in Example 2.1 illustrating different behaviour for the same set of subsystems depending on the applied switching sequence. The trajectory in dashed line results from the switching sequence in (2.5), the solid line corresponds to the switching sequence in (2.6). The initial state is $(1, 1)$ in both cases.

The above example demonstrates that a set of stable LTI systems can result in an asymptotically stable or an unstable switched system, depending on the set of switching signals associated with it. Hence the stability problem of switched systems break down to finding combinations of system matrices $\mathcal{A}$ and sets of admissible switching signal $\mathcal{S}$ that result in asymptotically stable switched systems. Naturally two questions arise:

(i) Given a set of switching signals $\mathcal{S}$, which class of system matrices $\mathcal{A}$ will result in asymptotical stable switched systems $\Sigma_{\mathcal{A},\mathcal{S}}$?

(ii) Given a set of matrices $\mathcal{A}$, find a set of switching signals $\mathcal{S}$, such that the switched system $\Sigma_{\mathcal{A},\mathcal{S}}$ is asymptotically stable.

We can identify four major problems of merit in the literature on stability of switched linear systems:

(i) **Arbitrary switching.** This problem is to identify sets of matrices $\mathcal{A}$ that result in exponentially stable switched systems with arbitrary switching signals, i.e. all switching signals described in Definition 2.1 result in asymptotically stable trajectories. In particular the system must be stable for constant switching signal $\sigma(t) = i$. Thus it is necessary that each of the constituent systems is a stable LTI system.

(ii) **Time-dependent switching.** This is the first of two problems that considers stability for a subset of switching signals. The switching signals are classified by the time-interval between two consecutive switching instances. The constituent systems are usually assumed stable. Hence sufficiently slow switching will result in asymptotic stable behaviour [LHM99]. The task is to find the minimum time between two consecutive switching instances to guarantee stability. The most prominent approaches are the dwell-time problem and the multiple-Lyapunov function approach.

(iii) **State-dependent switching.** This problem again, considers a subset of switching signals. In contrast to the previous problem, the switching signal is dependent on the state-vector. For every subsystem regions in the state-space are identified where they are allowed to be active while preserving stability of the switch system. The regions may overlap but have to cover the whole state-space as a union.

(iv) **Stabilising switching signals.** Here, the switched system is formed by unstable LTI systems. The goal is to construct switching signals, that result in stable trajectories. Such switching signals are commonly referred to as stabilising switching signals.

2.4 Lyapunov Theory

Lyapunov theory has played a key role in the stability analysis of linear and nonlinear system for a long time [Kal63, Rug96, Vid93]. It is therefore not surprising that many stability results for switched systems are based on Lyapunov Theory. In this section we briefly recall Lyapunov's stability theorem and define the common Lyapunov function for switched linear systems.

Common Lyapunov functions play an important role for the stability of switched systems. If there exists a common Lyapunov function for the constituent LTI systems then the switched system is exponentially stable for arbitrary switching signals. Conversely, a number of authors have shown that the existence of common Lyapunov function is also necessary for the exponential stability of the switched system.

2.4.1 Lyapunov functions

The idea behind Lyapunov's stability theory is as follows: assume there exists a positive definite function with a unique minimum at the equilibrium. One can think of such a function as a generalised description of the energy of the system. If we perturb the state from its equilibrium, the energy will initially rise. If the energy of the system constantly decreases along the solution of the autonomous system, it will eventually bring the state back to the equilibrium. Such functions are called *Lyapunov functions*. While Lyapunov theorems generalise to nonlinear systems and locally stable equilibria (see [Vid93] for a thorough discussion) we shall only state them in the form applicable to our system class.

Theorem 2.8 (Lyapunov stability) *The equilibrium $x = 0$ of the switched linear system* (2.2) *is uniformly asymptotically stable if there exists a continuous, differentiable and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ with*

$$\begin{aligned} V(0) &= 0 \qquad \textit{is a unique minimum,} && (2.7a) \\ V(x) &> 0 \qquad \textit{elsewhere} && (2.7b) \end{aligned}$$

such that the derivative along the solutions $x(t, t_0, x_0, \sigma)$ of (2.2) *is negative, i.e.*

$$\frac{d}{dt} V\big(x(t, t_0, x_0, \sigma)\big) < 0 \qquad \forall\ x_0 \neq 0,\ t \neq t_k \tag{2.8}$$

for all initial times t_0 and all switching signals $\sigma(\cdot) \in \mathcal{S}$, where t_k denotes the switching instances.

A function that satisfies the first two conditions (2.7) is sometimes referred to as *Lyapunov function candidate*.

The requirements that $V(x)$ is continuous and differentiable are classical and customary assumptions. In general, the function $V(x)$ does not need to be continuously differentiable as long as its decrease along the system trajectories can be guaranteed.

Note that the above Lyapunov theorem is applicable to switched systems with any class of switching signals. In the following we shall focus on applying Lyapunov theory to switched systems with arbitrary switching signals. If $V(x)$ is a Lyapunov function for the switched system (2.2) with arbitrary switching signals, $V(x)$ is certainly also a Lyapunov function for each of the constituent LTI systems Σ_{A_i}. Functions for which the properties (2.7) and (2.8) hold simultaneously for the systems Σ_{A_i} are called *common* Lyapunov functions.

Definition 2.9 (Common Lyapunov function) *The function $V : \mathbb{R}^n \to \mathbb{R}$ is a common Lyapunov function for the family of LTI systems Σ_{A_i}, $A_i \in \mathcal{A}$, if $V(x)$ is a Lyapunov function candidate and satisfies*

$$\nabla V(x) A_i x \quad < \quad 0 \qquad \forall\ A_i \in \mathcal{A},\ x \in \mathbb{R}^n \tag{2.9}$$

where ∇V denotes the gradient of V.

The existence of a common Lyapunov function for the constituent systems is sufficient for asymptotic stability of the switched system (2.2) with subsystems $\mathcal{A}$. To see this consider the following. Between the two consecutive switching instances t_k, t_{k+1} the switched system evolves according to $\Sigma_{A_{i_k}}$ for which $\nabla V(x) A_{i_k} x < 0$. At any switching instant the state is continuous and after switching into mode i_{k+1} we get $\nabla V(x) A_{i_{k+1}} x < 0$. Thus the derivative is always negative along the solutions of (2.2). A formal proof of the following theorem can be found in [Lib03].

Theorem 2.10 (Sufficient condition for asymptotic stability)
The switched linear system (2.2) *is asymptotically stable for arbitrary switching signals $\sigma(\cdot) \in \mathcal{S}$, if there exists a common Lyapunov function $V(x)$ for the constituent LTI systems Σ_{A_i}, $i \in \mathcal{I}$.*

Common Lyapunov functions are of great importance for establishing asymptotic stability of switched linear systems with arbitrary switching signals. We shall note some properties that will be used throughout this thesis.

(i) **Coordinate independence.**

Lemma 2.11 *Let $V(x)$ be a common Lyapunov function for the LTI systems Σ_{A_i} with $A_i \in \mathcal{A}$ and choose the co-ordinate transformation $\tilde{x} = Sx$ with $S \in \mathbb{R}^{n\times n}$, non-singular. Then $V(S^{-1}\tilde{x})$ is a common Lyapunov function for the transformed systems $\Sigma_{\tilde{A}_i}$ with $\tilde{A}_i = SA_iS^{-1}$.*

Proof. $V(S^{-1}\tilde{x})$ is a Lyapunov function for $\Sigma_{\tilde{A}_i}$ if and only if

$$\nabla V(S^{-1}\tilde{x})\, \tilde{A}_i \tilde{x} \quad < \quad 0$$

The gradient of $V(S^{-1}\tilde{x})$ with respect to $\tilde{x}$ is $\nabla V(S^{-1}\tilde{x})S^{-1}$. Thus

$$\nabla V(S^{-1}\tilde{x})\, S^{-1}\tilde{A}_i \tilde{x} \quad < \quad 0$$

Substituting $\tilde{A}_i = SA_iS^{-1}$ and $\tilde{x} = Sx$ yields

$$\nabla V(x)\, A_i x \quad < \quad 0$$

The latter holds according to the assumption that $V(x)$ is a Lyapunov function for Σ_{A_i}. □

(ii) **Convex combination.**

Lemma 2.12 *Let $V(x)$ be a common Lyapunov function for the LTI systems Σ_{A_i} with $A_i \in \mathcal{A}$. Then $V(x)$ is also a Lyapunov function for the LTI systems $\Sigma_{\bar{A}}$ with*

$$\bar{A} \quad = \quad \sum_{i=1}^{N} \alpha_i A_i \qquad \textit{with} \quad \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i = 1. \tag{2.10}$$

Proof. $V(x)$ is a Lyapunov function for the LTI system $\Sigma_{\bar{A}}$ if and only if

$$\nabla V(x) \sum_{i=1}^{N} \alpha_i A_i x \quad < \quad 0 \tag{2.11}$$

$$\Leftrightarrow \qquad \sum_{i=1}^{N} \alpha_i \nabla V(x) A_i x \quad < \quad 0\,.$$

Since $V(x)$ is a Lyapunov function for the LTI systems Σ_{A_i}, $\nabla V(x)A_ix$ is a negative scalar for all $A_i \in \mathcal{A}$. The convex combination of negative numbers yields a negative number. Hence (2.11) holds and $V(x)$ is a Lyapunov function for $\Sigma_{\bar{A}}$. □

(iii) **Robustness.**

Both exponential stability and the existence of a common Lyapunov function are robust properties in the sense that they are retained under sufficiently small parameter perturbations [SOCC00].

In particular the properties (i) and (ii) are used in many stability results for switched linear systems. Moreover, Lemma 2.12 implies that $V(x)$ is also a Lyapunov function for the (continuously) parameter-varying system

$$\dot{x}(t) = A(t)x(t)\,, \qquad A : \mathbb{R} \to \mathrm{co}\big\{A_1, \dots, A_N\big\}. \tag{2.12}$$

Thus stability results for switched linear systems guaranteeing the existence of a common Lyapunov function for the constituent systems, are also valid for the robust stability of the related parameter-varying system (2.12).

2.4.2 Converse Theorems

A number of converse Lyapunov theorems establish that the existence of a common Lyapunov function is also necessary for the asymptotic stability of the switched system (2.2) with arbitrary switching signals. While these theorems are not constructive, they provide an important theoretical basis for the development of analytic and theoretic stability results. Although, the results stated in this section are originally proven for differential inclusions and polysystems, we shall state them as they apply to switched linear systems of the form (2.2).

Dayawansa and Martin investigated in [DM99] the existence of common Lyapunov function for dynamical polysystems, a collection of smooth vector fields, similar to differential inclusions. In the context of switched linear systems they show that uniform asymptotic stability is equivalent to the existence a smooth common Lyapunov function.

Theorem 2.13 *The origin of the switched linear system* (2.2) *is stable if and only if there exists a C^∞ positive definite function $V(x)$ such that $\nabla V(x)A_i x < 0$ for all $A_i \in \mathcal{A}$.*[1]

In the same paper it is shown that the statement of the above theorem applies equivalently to the existence of a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$. Hence, a necessary and sufficient condition for uniform asymptotic stability of the switched linear system (2.2) is the existence of a continuously differentiable Lyapunov function as well as the existence of a smooth Lyapunov function.

[1]A function is C^∞ if it has continuous partial derivatives of all orders; a function is called C^1 if its first partial derivative is continuous.

In the context of differential inclusions Molchanov and Pyatnitskii investigated the existence of Lyapunov functions that are not necessarily continuously differentiable [MP86, MP89]. In this context, the usual derivative is replaced by

$$u(x) \quad = \quad \max_{i \in \mathcal{I}} \frac{\partial V(x)}{\partial \xi_i}$$

where $\xi_i = A_i x$ is the direction of the trajectory in the point x when the subsystem Σ_{A_i} is active, and

$$\frac{\partial V(x)}{\partial \xi} \quad = \quad \lim_{\Delta \to +0} \frac{V(x + \Delta\xi) - V(x)}{\Delta}$$

is the one-sided derivative. Then, $u(x)$ is the maximum over all one-sided derivatives of $V(x)$ at the point $x \in \mathbb{R}^n$ in the directions of $\xi_i = A_i x$.

With that, Molchanov and Pyatnitskii derive a number of important results in [MP86, MP89]. They show that a switched linear system is uniformly asymptotically stable if and only if there exists a common quasi-quadratic Lyapunov function for the constituent systems [MP89].

Theorem 2.14 *The origin of the switched linear system* (2.2) *is asymptotically stable if and only if there exists a strictly convex, homogeneous*[2] *of degree two Lyapunov function* $V(x)$ *of the form*

$$\begin{aligned} V(x) &= x^{\mathsf{T}} \mathcal{L}(x) x \,, \qquad \mathcal{L}(x) \in \mathbb{R}^{n \times n}, \\ \textit{with} \quad \mathcal{L}(x)^{\mathsf{T}} &= \mathcal{L}(x) \; = \; \mathcal{L}(ax) \,, \qquad a \in \mathbb{R} \\ V(0) &= 0 \end{aligned} \tag{2.13}$$

such that the derivative $u(x)$ *is strictly negative along all solutions* $x(t)$*, i.e.*

$$\max_{\xi_i = A_i x} \frac{\partial V(x)}{\partial \xi_i} \quad \leq \quad -\gamma \|x\|^2, \qquad \gamma > 0\,. \tag{2.14}$$

The function $V : \mathbb{R}^n \to \mathbb{R}$ is convex, but not always continuous differentiable. At every point x, $\mathcal{L}(x)$ is symmetric and $\mathcal{L}(\cdot)$ is radially constant. The description for $V(x)$ as a quasi-quadratic, stems from the form of (2.13). In case that $\mathcal{L}(x)$ is piecewise constant a piecewise quadratic Lyapunov function is obtained; if $\mathcal{L}(x)$ is constant over the whole state-space, $V(x)$ is a common quadratic Lyapunov function.

Consider the piecewise linear function

$$V_m(x) \quad = \quad \|Lx\|_\infty \tag{2.15}$$

[2] A function $V : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree p if $V(ax) = a^p V(x)$ for all $a > 0$ and $x \in \mathbb{R}^n$.

where $L \in \mathbb{R}^{m \times n}$ and $rank\{L\} = n \leq m$. $V_m(x)$ satisfies condition (2.7), thus $V_m(x)$ is a Lyapunov function candidate. The row-vectors of L are the normals to the faces of centrally symmetric polyhedrons. Therefore Lyapunov functions of the form (2.15) are sometimes also referred to as polyhedral Lyapunov functions.

The level surface of (2.13) can be approximated by the surface of such convex polyhedron. It is shown in [MP86] that by choosing m large enough, it is always possible to find a function $V_m(x)$ of the form (2.15) such that the condition (2.14) is satisfied.

With $V_m(x)$ we can associate a piecewise *quadratic* Lyapunov function

$$V(x) \quad = \quad \|Lx\|_\infty^2 \tag{2.16}$$

which also defines a necessary and sufficient condition for the asymptotic stability of the switched linear system (2.2).

Theorem 2.15 *[MP86] The origin of the switched linear system* (2.2) *is asymptotically stable if and only if there exists a piecewise linear Lyapunov function* (2.15) *(piecewise quadratic Lyapunov function* (2.16)*, respectively) whose derivative along all solutions is negative as in* (2.14)*.*

With these results, general conditions for the existence of common piecewise linear and piecewise quadratic Lyapunov functions is established. These converse theorems provide a theoretical justification for the search of Lyapunov function conditions as described in the following sections and for much of the work in this thesis.

Similar results in the context of discrete-time switched systems have been obtained by Brayton and Tong in [BT79]. For background to that material and the relation to the absolute stability problem refer to [MP86, Pya70]. Further results on general classes of Lyapunov functions and their existence for classes of uncertain systems can be found in [Bla94, Bla95, BM99].

2.5 Stability results for arbitrary switching

The results described in the previous section provide general necessary and sufficient conditions for the asymptotic stability of switched linear systems for arbitrary switching signals. However, in order to apply these results, we need to develop methods to find such common Lyapunov function or derive conditions that guarantee its existence. Since it is certainly not feasible to randomly test all possible Lyapunov function candidates, stability results are commonly restricted to a specific class of Lyapunov function candidates. In this section we describe some of the methods developed in the past.

2.5.1 Common quadratic Lyapunov functions

A necessary condition for the asymptotic stability of the switched system (2.2) with arbitrary switching signals is that the constituent LTI systems Σ_{A_i}, $i \in \mathcal{I}$ are asymptotically stable. It is well known that the existence of a *quadratic* Lyapunov function is necessary and sufficient for asymptotic stability of LTI systems. A natural question is therefore, under which conditions a number of LTI systems share a *common* quadratic Lyapunov function. In this section we collect conditions for the existence of such functions.

Given the quadratic form

$$V(x) = x^{\mathsf{T}} P x \qquad \text{with} \quad P = P^{\mathsf{T}} > 0 \tag{2.17}$$

it is readily verified, that such function satisfies the conditions (2.7) for a Lyapunov function candidate.

We speak of $V(x)$ as a common quadratic Lyapunov function (CQLF) for the LTI systems Σ_{A_i} with $A_i \in \mathcal{A}$, if (2.17) satisfies simultaneously

$$A_i^{\mathsf{T}} P + P A_i = -Q_i, \quad \text{with} \tag{2.18}$$

$$Q_i = Q_i^{\mathsf{T}} > 0 \quad \text{for all } i \in \{1, \ldots, N\}\,. \tag{2.19}$$

The existence of such a common Lyapunov function is sufficient but in general not necessary for the asymptotic stability of the switched linear system (2.2) [DM99]. However, it is possible to identify system classes for which the existence of a CQLF is also necessary [Mas04].

Triangular system matrices

There is number of classes of switched systems for which the existence of a CQLF is readily given by the structural properties of the system matrices $A_i \in \mathcal{A}$. A relatively simple example for such a class are switched systems where the system matrices A_i,

$i \in \mathcal{I}$ are Hurwitz and symmetric [CL97]. In this case the matrices A_i are negative definite and hence the identity matrix I is a common Lyapunov function for the constituent LTI systems.

For switched systems with two constituent system we note the following result from [BBP78] that proves useful deriving stability conditions for pairs of LTI systems.

Theorem 2.16 *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix. Then any quadratic Lyapunov function $V(x)$ for the LTI system Σ_A is also a Lyapunov function for $\Sigma_{A^{-1}}$.*

The CQLF existence for sets of matrices that can be simultaneously transformed into upper triangular form is subject of various publications. It has been shown by several authors that a set of matrices $A_i \in \mathcal{A}$ that are in upper triangular form and Hurwitz have a CQLF. Hence by Lemma 2.11, so does any set of matrices that can be simultaneously transformed into upper triangular form by similarity transformation (see e.g. [MMK96, SN98b]). The version of the theorem stated here is from [SN98a].

Theorem 2.17 *A sufficient condition for the Hurwitz matrices $A_i \in \mathcal{A}$ sharing a common quadratic Lyapunov function is that there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$ such that $TA_iT^{-1} \in \mathbb{C}^{n\times n}$ is upper triangular for all $i \in \{1,\ldots,N\}$.*

The proof of this theorem uses on the fact that systems of this form can be represented as cascades of first order or certain (very benign) second order subsystems, which again can be shown to be asymptotically stable.

The strength of Theorem 2.17 is that it is applicable to switched systems with any finite number of subsystems of arbitrary system order. However, it is not easy to determine whether a set of matrices is simultaneously triangularisable. Theorem 2.17 includes a much earlier results on the CQLF problem. Narendra and Balakrishnan established that a set of stable LTI systems with commuting system matrices, i.e. $A_iA_j = A_jA_i \ \forall\, i,j \in \{1,\ldots,N\}$ has a common quadratic Lyapunov function [NB94]. However, it is a well known result of linear algebra that a commuting family of real matrices can be simultaneously triangularised by a real orthogonal matrix [HJ85]. A number of further conditions on simultaneously triangularisable systems can be found in [Laf78].

In [LHM99, AL01] the problem is related to the solvability of Lie algebras. The Lie algebra conditions can be considered as an extension of the commuting matrices results above by linking the commutators $[A_i, A_j]$ to the existence of a CQLF. It turns out, that matrices for which the Lie algebra is solvable are simultaneously triangularisable.

A further drawback is that the property of simultaneous triangularisation is not robust under parameter perturbations (see [SOCC00] for an example). A robustness analysis of Theorem 2.17 can be found in [MMK97]. The approach here is to bound the maximum allowable perturbations of the matrix parameters from a nominal (triangularisable) set of matrices, while guaranteeing the existence of a CQLF. Other approaches extend the results by relaxing the simultaneous triangularisabilty condition. In [SOC01, SOC02] conditions are derived where the triangular property only needs to apply for any *pair* of matrices in $\mathcal{A}$. Here asymptotic stability of the switched system is guaranteed although there might not exist a CQLF *for all* constituent systems. However, all those conditions suffer from a more or less strong restriction on the structure of the system matrices A_i.

Matrix pencil conditions

The convexity property of common Lyapunov functions (Lemma 2.12) provides a powerful tool for the stability analysis of switched linear systems. Recall that $V(x)$ is a common Lyapunov function for the LTI systems Σ_{A_i}, $i \in \mathcal{I}$ only if it is also a Lyapunov function for all LTI systems $\Sigma_{\bar{A}}$ with

$$\bar{A} \quad = \quad \sum_{i=1}^{N} \alpha_i A_i \qquad \text{with} \quad \alpha_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i = 1.$$

It is well known that there exists a Lyapunov function for the LTI system $\Sigma_{\bar{A}}$ if and only if the eigenvalues of $\bar{A}$ lie in the open left half of the complex plane. If the spectrum of $co\{A_1, \ldots, A_N\}$ has an eigenvalue with positive real part, it follows that there exists an LTI system with system matrix $\bar{A} \in co\{A_1, \ldots, A_N\}$ for which no Lyapunov function exists. Then it follows from the converse theorems in Section 2.4.2 that the switched system (2.2) with $\mathcal{A} = \{A_1, \ldots, A_N\}$ is not asymptotically stable for arbitrary switching signals. In fact, it has been shown in [SOCC00] that in such case there exists a destabilising switching sequence.

There are a number of stability conditions for switched systems that are formulated in terms of the spectrum of the convex combination of the constituent system matrices for switched systems that consist of two subsystems. To comply with standard terminology of the literature we shall refer to the convex combination $\alpha A_1 + (1-\alpha)A_2$,

$\alpha \in [0,1]$ as the matrix pencil $\sigma_\alpha[A_1, A_2]$. There are two such matrix pencil results concerning the existence of a CQLF for second order systems.

Theorem 2.18 *[SN97] A sufficient condition for the existence of a CQLF for the switched system* (2.2) *with* $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{2\times 2}$ *is that the eigenvalues of the matrix pencil* $\sigma_\alpha[A_1, A_2]$ *are negative and real for all* $\alpha \in [0,1]$.

Of course, the condition of this theorem requires that the matrices A_i, $i = 1, 2$ are Hurwitz and have real eigenvalues. This restriction can be relaxed adding a further condition which also yields necessity for the existence of a CQLF [SN99]:

Theorem 2.19 *A necessary and sufficient condition for the existence of a CQLF for the switched system* (2.2) *with* $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{2\times 2}$ *is that the eigenvalues of the matrix pencils* $\sigma_\alpha[A_1, A_2]$ *and* $\sigma_\alpha[A_1, A_2^{-1}]$ *are in the open left complex plane for all* $\alpha \in [0,1]$.

This theorem is inspired by Theorem 2.16 and its implications regarding the spectrum of the matrix pencil. Theorem 2.16 provides that any quadratic Lyapunov function $V(x)$ for Σ_A, with $A \in \mathbb{R}^{n\times n}$ is also a Lyapunov function for $\Sigma_{A^{-1}}$. Hence, there exists a CQLF for the systems Σ_{A_1} and Σ_{A_2} if and only if there exists a CQLF for Σ_{A_1} and $\Sigma_{A_2^{-1}}$.

This reasoning generalises to systems with N subsystem. Consider the set $\mathcal{A}_k$ obtained by substituting any k matrices $A_i \in \mathcal{A}$ by their inverse A_i^{-1}. It follows that it is necessary for the existence of a CQLF for the LTI systems Σ_{A_i}, $A_i \in \mathcal{A}$, that all matrices in the convex hull $co\big(\mathcal{A}_k\big)$ are Hurwitz for all $0 \le k \le N$.

In [SN02] a relation of the eigenvalues of the matrix pencil $\sigma_\alpha[A_1, A_2]$ for all $\alpha \in [0,1]$ and the eigenvalues of the product $A_2^{-1}A_1$ is established.

Lemma 2.20 *Let* $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *be Hurwitz matrices. The eigenvalues of the matrix-pencil* $\sigma_\alpha[A_1, A_2]$ *are in the open left half-plane for all* $\alpha \in [0,1]$ *if and only if the matrix product* $A_2^{-1}A_1$ *has no negative, real eigenvalue.*

Using this lemma the matrix pencil condition of Theorem 2.19 reduces to a simple eigenvalue test of the matrix products $A_2^{-1}A_1$ and A_1A_2 when $A_1, A_2 \in \mathbb{R}^{2\times 2}$. This result has been recently extended to a class of n-th order systems [SMCC04]:

Theorem 2.21 *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Hurwitz matrices where* $\operatorname{rank}\{A_1 - A_2\} = 1$. *There exists a CQLF for the LTI systems Σ_{A_1} and Σ_{A_2} if and only if the matrix product A_1A_2 has no negative, real eigenvalues.*

Linear matrix inequalities

The problem of finding a CQLF for a set of LTI systems can be formulated as a feasibility problem of linear matrix inequalities [BEFB94, EGN00].

Recall that $V(x) = x^{\mathsf{T}}Px$ is a CQLF for the systems Σ_{A_i}, if P is a symmetric, positive definite matrix and satisfies inequality (2.18) simultaneously for all $A_i \in \mathcal{A}$. These conditions define a system of linear matrix inequalities (LMIs) in P, namely

$$P \quad = \quad P^{\mathsf{T}} \quad > \quad 0 \tag{2.20a}$$

$$(A_i^{\mathsf{T}}P + PA_i) \quad < \quad 0 \quad \text{for} \quad i \in \{1, \ldots, N\}\,. \tag{2.20b}$$

The system of LMIs (2.20) is said to be *feasible* if a solution P exists. Therefore, the problem reduces to the convex optimisation problem of checking whether the LMIs (2.20) are feasible. Due to the powerful convex optimisation algorithms developed over the last two decades, such problems can be solved with great efficiency. Implementation of such algorithms in software packages like the LMI toolbox for MATLAB [GNLC95] have made the approach widely used.

The great advantage of the LMI approach is that it is applicable to switched systems of any order and with any number of subsystems; only restricted by computational capacities. However due to its numerical nature, little insights are provided into why a CQLF may or may not exist. Moreover, it has been shown that it is not hard to construct examples for which the most commonly used LMI toolbox for MATLAB fails to give the appropriate answer [MSL01].

2.5.2 Common piecewise linear Lyapunov functions

The converse Theorem 2.15 provides that there exists a common piecewise linear Lyapunov function (PLF) if and only if the switched linear system (2.2) is asymptotically stable. It is therefore reasonable to search for simple conditions that guarantee the existence of such type of common Lyapunov function for a set of LTI systems.

Piecewise linear functions are commonly defined in terms of the l_1 or the l_∞ norm.

We shall use the definition in [MP89]:

$$V(x) = \|Lx\|_\infty \tag{2.21}$$

where $L \in \mathbb{R}^{m\times n}, m \geq n$ has full rank n. Such functions are radially unbounded, have a unique minimum and the onesided derivative exists [MP89, KAS92].

For a common piecewise linear Lyapunov function for a set of LTI systems the following condition is obtained in [MP89]:

Theorem 2.22 *The function $V(x) = \|Lx\|_\infty$ is a common piecewise linear Lyapunov function for the LTI systems Σ_{A_i} with $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}$ if and only if there exist $Q_i \in \mathbb{R}^{m\times m}$, $i \in \mathcal{I}$ such that*

$$q_{kk} + \sum_{\substack{l=1\\ l\neq k}}^{N} \|q_{lk}\| < 0 \tag{2.22}$$

and

$$LA_i - Q_iL = 0 \tag{2.23}$$

Remark: *This theorem is the special case for $p = \infty$ of the result in [KAS92] for general Lyapunov functions based on p-norms, $V(x) = \|Lx\|_p$.*

The approach of utilising piecewise linear Lyapunov functions can be tracked back to the sixties with a series of papers by Rosenbrock [Ros62] and Weissenberger [Wei69] on Lur'e type systems. But powerful algebraic tools for the existence of piecewise linear Lyapunov functions remain scarce. One reason for this might be due to the open problem of deciding on the number of faces m that are needed to find a piecewise linear Lyapunov function for a given system.

Regarding LTI systems, this problem has been recently addressed in [BP01] and [Bob02]. In this work, the authors relate the number of faces of the PLF (2.21) to the location of the spectrum of the system matrix A. Let $K(m)$ denote the sector of the complex plane whose bisector is the negative real axis and whose angle is $\left(1 - \frac{2}{m}\right)\pi$.

Theorem 2.23 *[BP01] Let $V(x)$ with $L \in \mathbb{R}^{m\times n}$ be a piecewise linear Lyapunov function (2.21) for the LTI system Σ_A with $A \in \mathbb{R}^{n\times n}$ and the eigenvalues of A are distinct. Then the following holds:*

$$\sigma(A) \subset K(m).$$

Of particular interest is the minimum number of faces for which there exists a PLF for a given LTI system Σ_A. We shall denote this number as $\nu(\Sigma_A)$. For LTI systems with real eigenvalues the minimum number of faces $\nu(\Sigma_A)$ can be directly computed [BP01]:

Theorem 2.24 *Let Σ_A be a LTI system where $A \in \mathbb{R}^{n\times n}$ has distinct real eigenvalues. Then $\nu(\Sigma_A) = n + 1$ and the spectrum of A lies in the sector $K(\nu(\Sigma_A))$.*

For systems with non-real eigenvalues the minimum number of faces $\nu(\Sigma_A)$ is still unknown. However, an upper bound for $\nu(\Sigma_A)$ is given in [Bob02].

The above theorems can serve as a starting point for the derivation of conditions for the existence of a common PLF for a set of LTI systems. Using the convexity property of Lemma 2.12 we can immediately find necessary conditions in terms of the eigenvalues of the matrices of the convex hull $co(\mathcal{A})$.

Theorem 2.25 *Consider the LTI systems Σ_{A_i}, $A_i \in \mathcal{A} \subset \mathbb{R}^{n\times n}$ where the eigenvalues of A_i are distinct for all i. Let $V(x) = \|Lx\|_\infty$ with $L \in \mathbb{R}^{m\times n}$ be a common PLF for the LTI systems Σ_{A_i}, $A_i \in \mathcal{A}$, then*

$$\bigcup_{A \in co(\mathcal{A})} \sigma(A) \subset K(m).$$

Proof. The proof follows directly from Theorem 2.23 and the convexity property of the common Lyapunov function in Lemma 2.12. □

Most results for the existence of a common PLF are of numerical nature. As for the search of common quadratic Lyapunov functions, optimisation algorithms are used to find matrices L and Q_i that satisfy the conditions (2.22) and (2.23). However, the search for a common PLF is more complex since the size of L is unknown. Common algorithms starting with an initial set of vertices (or alternatively faces) of a polyhedron, check whether the conditions (2.22) and (2.23) are satisfied for the constituent systems and refine the shape (and possibly the number of vertices) until an appropriate function is found.

Brayton and Tong were one of the first to suggest an algorithm to successively construct a PLF for discrete-time systems [BT79, BT80]. The majority of methods for calculating PLFs are similar and mainly seek to find algorithms that converge more reliably and efficiently. Powerful approaches using linear programming for fast optimisation and various refining methods have been suggested by Polański [Pol95, Pol97, Pol00] and Yfoulis *et al.* [YMWP99, YMW02, YS04] . However, the computational effort still increases immensely with the system order n. Consequently, the algorithms are currently only feasible for systems of order three, maybe four.

2.6 Stability results for constrained switching

The results we discussed so far consider asymptotic stability of switched linear systems with arbitrary switching signals. However, in many practical applications some switching signals can be discarded. For example, this could be due to bandwidth limitations of the hardware that do not allow switching above a certain frequency or due to other design specifications that exclude certain switching sequences. In these cases common Lyapunov function approaches can lead to conservative results since they even provide for dynamics caused by switching signals that are not admissible for the system. Even if a common Lyapunov function fails to exist, it might still be possible to establish stability for a (sub-)class of switching signals. This is in particular important for systems with unstable constituent systems.

In this section we describe a number of stability conditions that account for particular properties of the switching signal. The first approach bases upon the idea of monitoring the evolution of a set of Lyapunov functions (Multiple Lyapunov function approach) which yields indirectly a stable switching signal. The second guarantees stability by explicit calculation of a minimum time between consecutive switches (dwell-time approach).

In the third part of this section we briefly describe a number of results obtained for piecewise linear systems where the switching action depends on the current state of the system. This approach leads to a partitioned state-space where switching occurs along specific switching surfaces. These partitions can be used to construct piecewise Lyapunov functions for the switched system.

2.6.1 Multiple Lyapunov function

The key idea of the multiple Lyapunov function approach is to employ a number of different Lyapunov functions (commonly one for each subsystem) instead of a single

Lyapunov function for all constituent systems. The idea is to derive conditions on the switching signal such that the switched system is guaranteed to be stable. In other words, given a set of system matrices $\mathcal{A}$ we identify a class of switching signals for which an asymptotically stable switched system is obtained.

Roughly speaking the idea is that we only allow to switch back to a particular mode i if the value of the associated Lyapunov function $V_i(x)$ has decreased since we left that mode the last time. This idea should probably be credited to Peleties and DeCarlo who showed that asymptotic stability is achieved by switching in such fashion [PD91].[3] However, several authors have modified and extended this approach over the last ten years. Branicky formulated this approach as a general concept for stability analysis and established results that are also applicable for switched systems with nonlinear subsystems in [Bra98]. In this section we shall describe this result in terms of the switched linear system (2.2) to illustrate the main ideas behind this method. For a thorough overview on multiple Lyapunov functions see [DBPL00] and reference therein.

Given the switching system (2.2) we associate Lyapunov functions $V_i(x) \in \mathcal{V}$ for each subsystem $A_i \in \mathcal{A}$. While each V_i decreases when the i^{th} subsystem is active, any other function V_j, $j \neq i$, may increase (c.f. Figure 2.4). Consider now the values of V_i at the beginning of each interval. If the sequence of these initial values is decreasing for all modes i, the system is asymptotically stable.

Theorem 2.26 *Let $\mathcal{A}$ be a set of stable system matrices $A_i \in \mathcal{A}$, let $V_i \in \mathcal{V}$ be a Lyapunov functions for the LTI system Σ_{A_i} for all $i \in \mathcal{I}$ and let $\sigma(\cdot) \in \mathcal{S}$ be a switching signal. Consider the pairs of switching instances $(t_j, t_l)_i$ for the mode i, with $j < l$ such that $\sigma(t_j) = \sigma(t_l) = i$ and $\sigma(t_k) \neq i$ for $t_i < t_k < t_l$. The switched linear system $\Sigma_{\mathcal{A},\mathcal{S}}$ is asymptotically stable if there exist a $\gamma > 0$ such that*

$$V_i(x(t_l)) - V_i(x(t_j)) \quad \leq \quad -\gamma \|x(t_j)\|^2 \tag{2.24}$$

for all pairs $(t_j, t_l)_i$ and all $\sigma(\cdot) \in \mathcal{S}$.

Note, that if (2.24) holds for all switching signals defined in Definition 2.1 then according to the converse Theorems 2.14 and 2.15 there exists also a common Lyapunov

[3] In the original paper the authors consider systems that switch along switching surfaces in the state-space. This approach is discussed in Section 2.6.3.

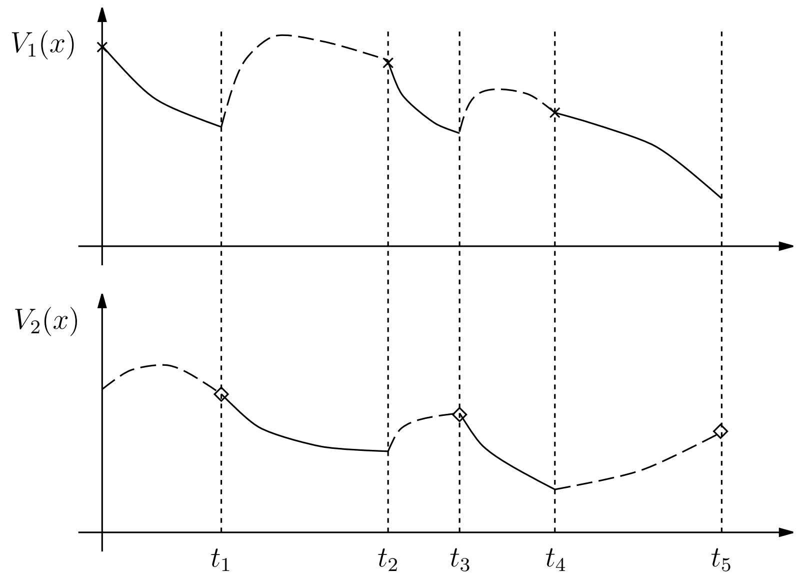


Figure 2.4: Multiple Lyapunov functions. Evolution of the Lyapunov functions V_1 and V_2. The solid lines indicate that the respective subsystem is switched in and therefore V_i decrease. When the other system is active, the value may increase (dashed lines). If the sequences of initial values '×' and '$\diamond$' constantly decrease, the switched system is asymptotically stable.

function for all subsystems. However, Theorem 2.26 does not rely on the explicit construction of this common Lyapunov function.

If a common Lyapunov function fails to exist, we might be still able to construct stable switching signals using multiple Lyapunov functions. However, the resulting set of switching signals $\mathcal{S}$ depends on the initial choice of Lyapunov functions V_i. Therefore, an unfortunate choice of Lyapunov functions may lead to conservative switching rules or even no result. Further, in order to apply Theorem 2.26 we need the values of the respective Lyapunov functions $V_i(x)$ at the switching instances. In general this requires the knowledge of the state at these times, whereas classical Lyapunov stability approaches do not require any knowledge about the solutions.

The multiple Lyapunov function approach is not dependent on the system order or number of subsystems. It can even be applied to heterogenous hybrid systems (where the structure of the subsystems is not homogeneous) and to hybrid systems that switch between nonlinear vector fields $\dot{x} = f_i(x)$ as Branicky proves in a first generalisation of the multiple Lyapunov function approach in [Bra94, Bra98].

The original theorem in [PD91] applies multiple Lyapunov functions to switched systems with unstable subsystems. This requires a relaxation on the properties of the function V_i. Instead of demanding that their derivatives along the solutions of the subsystem A_i is negative for the whole state space, we can only require that this holds for certain regions $\Omega_i \subset \mathbb{R}^{n\times n}$. In this case the functions V_i are often call Lyapunov-like functions. This allows more freedom of choosing appropriate functions V_i, however, the associated system A_i is only allowed to be active if $x \in \Omega_i$. Obviously, the union of all regions $\bigcup_i \Omega_i$ has to cover the whole state space (c.f. Section 2.6.3 on state dependent switching below).

In a more general framework, but also applicable to our case, Ye *et al.* relax this condition further [YMH98, Mic99]. Here, the value of V_i may increase in a bounded fashion, even if the respective system A_i is active, i.e. A_i may be switched in even if $x \notin \Omega_i$. Asymptotic stability can be shown if the initial values of $V_i(x(t_k))$ form a decreasing sequence for each mode.

2.6.2 Dwell-time approach

An alternative approach to determine a set of switching signals $\mathcal{S}$ that yields asymptotically stable switched systems $\Sigma_{\mathcal{A},\mathcal{S}}$ is to restrict the minimum time interval between two consecutive switching instances. This time constant $\tau_D \leq t_{i+1} - t_i \; \forall \; i \in \mathcal{I}$ has been termed dwell-time[4]. When the constituent subsystems Σ_{A_i} are asymptotically stable, sufficiently slow switching will yield stability.

For switched linear systems this dwell-time can be explicitly calculated [Mor96].

Theorem 2.27 *Let $\mathcal{S}_{\tau_D}$ denote the set of switching signals with a dwell-time τ_D. Then, the switched linear system* (2.2) *is asymptotically stable for the switching signals $\sigma(\cdot) \in \mathcal{S}_{\tau_D}$ if*

$$\tau_D \quad > \quad \sup_i \left(\frac{a_i}{\lambda_i}\right)$$

with $a_i \geq 0$ and $\lambda_i > 0$ such that

$$\left\| e^{A_i t} \right\| \quad < \quad e^{a_i - \lambda_i t} \qquad \forall \; t > 0.$$

Choosing the dwell-time as in the above theorem, ensures that the norm of the transition matrix $\Phi(t, t_0, x_0, \sigma)$ of the switched system is bounded by the transition matrix

[4]The term is chosen to suggest that the switching signal "dwells" on each of its values for at least τ_D time-units

of the subsystem Σ_{A_i} with the smallest decay rate.

This result was extended by Hespanha and Morse in [HM99], where the *average* dwell-time $\bar{\tau}_D$ is introduced. Here single switching intervals in $\sigma(\cdot)$ are allowed to be smaller than $\bar{\tau}_D$ provided that the average of all switching intervals in $\sigma(\cdot)$ is no smaller than $\bar{\tau}_D$. Discarding the first N_0 switchings, we request that the number of switchings between any $t_2 > t_1 \geq t_0$ does not exceed $\frac{t_2-t_1}{\bar{\tau}_D}$ afterwards. The set of such switching signals is denoted with $\mathcal{S}[\bar{\tau}_D, N_0]$.

Theorem 2.28 *Given the set of system matrices $A_i \in \mathcal{A}$, $i \in \mathcal{I}$ and a positive constant λ_0 such that $A_i + \lambda_0 I$ is asymptotically stable for all $i \in \mathcal{I}$, then, for any chosen $\lambda \in [0, \lambda_0)$, there is a finite constant $\bar{\tau}_D^*$ such that* (2.2) *is asymptotically stable for all switching signals $\sigma(\cdot) \in \mathcal{S}[\bar{\tau}_D, N_0]$ if $\bar{\tau}_D \geq \bar{\tau}_D^*$.*

This average dwell-time $\bar{\tau}_D^*$ can be calculated involving similar parameters of the subsystems as in Theorem 2.27.

The work of Zhai *et al.* [ZHYM00a] modifies this result such that the lowest average dwell-time $\bar{\tau}_D^*$ ensures that the switched system achieves a chosen L_2 gain. The above results can be extended such that stable and unstable subsystems Σ_{A_i} are allowed (e.g. [ZHYM00b, Yed01]).

2.6.3 State-dependent switching

In the previous subsections we discussed approaches where the stability conditions impose restrictions on the switching instances. Another approach to constrained switching is state-dependent switching, where switching occurs whenever the state-trajectory crosses a pre-defined switching surface. The switching signal in (2.2) is then formally defined by $\sigma : \mathbb{R}^{n\times n} \to \mathcal{I}$. However, for state-dependent switching it is common to characterise the switching by a partitioned state-space. The partitions describe regions Ω_i in the state-space where a certain subsystem Σ_{A_i} is active.

There is a number of problems that can be considered using a partitioned state-space. Typical examples include the stabilisation of switched systems with unstable subsystems and optimal control for constraint systems. In this section we shall only outline some of the approaches to illustrate the main ideas.

State-dependent switching appears in some of the earliest papers on the design of switching systems [PT73, PD91, PD92]. One way to establish stability is to find a so-called Lyapunov-like function whose derivative along the solutions of the subsystems is only negative in the regions where the respective subsystems are actually active.

As in the multiple Lyapunov function approach we define a family of function $V_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \mathcal{I}$, each associated with one of the subsystems Σ_{A_i}. A Lyapunov-like function for the system Σ_{A_i} with the equilibrium $x_0 \in \Omega_i \subset \mathbb{R}^{n\times n}$ is a real-valued function that satisfies the conditions for a Lyapunov function for Σ_{A_i} in Ω_i, namely

$$
\begin{aligned}
V_i(0) &= 0 && \text{(2.25a)}\\
V_i(x) &> 0 && \text{(2.25b)}\\
\frac{d}{dt}\, V\big(x(t,t_0,x_0,\sigma)\big) &< 0 && \text{(2.25c)}
\end{aligned}
$$

for all $x \in \Omega_i$. In other words, outside the region Ω_i none of the above conditions need to hold.

There are several approaches to establish stability for a switched system with such a partitioned state space. One such approach is a version of the multiple Lyapunov function approach as described in Section 2.6.1 [PD91, PD92].

When using quadratic Lyapunov-like functions V_i, the problem of finding the functions V_i that satisfy the relaxed conditions (2.25) can be formulated as LMIs. Consider the simple case where the same Lyapunov-like functions $V_i(x) = x^\top P x$, $i \in \mathcal{I}$ is used in all regions Ω_i. For each region Ω_i we require that $x^\top(A_i^\top P + PA_i)x$ is negative for $x \in \Omega_i$. To account for this relaxed requirement a method known as S-procedure is applied. Construct matrices S_i such that $x^\top S x \geq 0$ for $x \in \Omega_i$. Then we obtain the following relaxed stability condition

$$
A_i^\top P + PA_i + S_i \quad < \quad 0 \tag{2.26}
$$

for all $i \in \mathcal{I}$. Since $x^\top S x \geq 0$ for $x \in \Omega_i$, then $x^\top(A_i^\top P + PA_i)x < 0$ is satisfied when $x \in \Omega_i$. However, $x^\top S_i x$ is allowed to be negative outside Ω_i and hence allows for $x^\top(A_i^\top P + PA_i)x > 0$ in those regions. Condition (2.26) defines a system of LMIs in the variable P and therefore we can apply the methods of Section 2.5.1 to test its feasibility.

In [JR98] this approach is extended to formulate LMIs where the Lyapunov-like functions are quadratic functions $V_i(x) = x^\top P_i x$, but different for each region. By formulating further constraints it is ensured that the Lyapunov-like functions are continuously joined together at the boundary of the regions, i.e. $V_i(x) = V_j(x)$ for $x \in \Omega_i \cap \Omega_j$.

Thus when the LMI is feasible we obtain a piecewise quadratic Lyapunov function for the switched system. An example of such a Lyapunov function is shown in Figure 2.5.

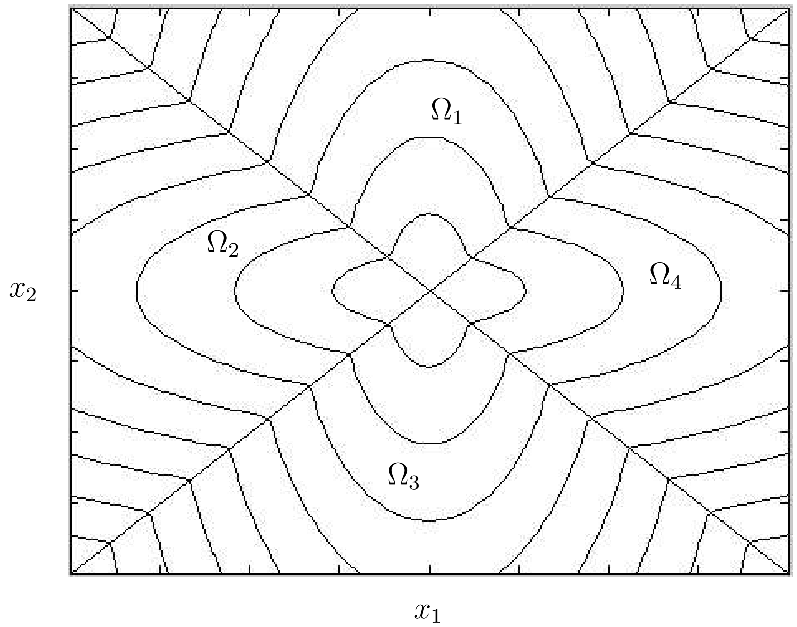


Figure 2.5: Level curves of a piecewise quadratic Lyapunov function constructed for a partitioned state-space using the LMI-approach in [JR98].

2.7 Conclusions

In this chapter we defined the system class and the stability notions that are the subject of this thesis. The stability analysis of switched linear systems is much more complex than the stability of LTI systems and raises a number of stability problems. Despite the large number of available stability results, there are still many questions unresolved. In particular there is a pressing need for results that can easily be used as a basis to develop powerful design methods for switched linear systems.

While the theoretical stability results are valid for a wide range of system classes and supply an important basis for their stability analysis, they are usually non-constructive and therefore not immediately applicable for a given control-problem. In contrast to this, the numerical tools provide a direct answer to the stability problem for a given system. However, they are lacking any guideline for the control design and do not provide any insight into the stability or instability mechanisms. Moreover, they are either restricted to quadratic stability (LMI approach) or are only applicable for systems of relatively low order.

Analytical stability tools have the potential to bridge the gap between stability theory and practical control design. However, most available results impose considerable restrictions on the system class and commonly only consider quadratic stability which can lead to conservative results. The objective in this thesis is twofold. Firstly, we aim to derive stability conditions that provide some assistance for the control-design for switched linear systems; and secondly, we use non-quadratic approaches in order to establish stability for a larger set of systems than purely quadratic approaches can provide.

Chapter 3

The 45° Criterion

We consider asymptotic stability of the class of switched linear systems $\Sigma_{\mathcal{A}}$ consisting of two subsystems $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{2\times 2}$ where arbitrary switching signals are admissible. It is shown that the switched linear system $\Sigma_{\mathcal{A}}$ is asymptotically stable if the spectrum of $\alpha A_1 + (1-\alpha)A_2$ lies within a certain region of the complex plane for all $\alpha \in [0,1]$. It is demonstrated that the stability result extends upon quadratic stability by using a new type of piecewise linear Lyapunov function. The existence of a common quadratic Lyapunov function and the common piecewise linear Lyapunov function can be related to properties of the spectrum of the convex combination for different subclasses of the second-order switched system.

3.1 Introductory remarks

In this chapter we consider the class of switched linear systems with two subsystems Σ_{A_1} and Σ_{A_2} where $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Thus the system dynamics are given by

$$\dot{x}(t) \quad = \quad A(t)x(t) \qquad A(t) \in \mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{2\times 2} \tag{3.1}$$

while the set of switching signals $\mathcal{S}$ contains all functions as in Definition 2.1, i.e. we consider arbitrary switching between the matrices A_1 and A_2. Therefore we shall omit the set of switching signals for the remainder of this chapter and simply refer to (3.1) as the switched system $\Sigma_{\mathcal{A}}$. In this chapter we derive conditions on the matrices A_1 and A_2 that guarantee asymptotic stability of the switched system $\Sigma_{\mathcal{A}}$ by the existence of a common Lyapunov function.

The converse Lyapunov theorems in Section 2.4 establish that a necessary and sufficient condition for asymptotic stability of the switched system (3.1) is that there exists a common Lyapunov function for A_1 and A_2. Of course this requires, that the constituent systems are stable LTI system, i.e. A_1 and A_2 are Hurwitz matrices. For LTI systems the Hurwitz property of the system matrix is necessary and sufficient for both, asymptotic stability and the existence of a quadratic Lyapunov function. However, this does not guarantee stability for the switched system as illustrated in Example 2.1. Yet, the existence of a *common* quadratic Lyapunov function (CQLF) ensures asymptotic stability of the switched system.

Even though, common quadratic Lyapunov functions proved to be very useful for establishing stability, there are cases where such approach can lead to conservative results [Bro65, DM99, SN98b]. In particular, the example in [DM99] demonstrates that switched linear systems can be asymptotically stable for arbitrary switching even if the constituent systems have no common *quadratic* Lyapunov function. The work of this chapter aims to extend the class of systems for which asymptotic stability can be established by introducing a new type of Lyapunov function.

The stability condition derived in this chapter is formulated in terms of the eigenvalues of the convex combination $\alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$. Although this approach has been considered before for the existence of common quadratic Lyapunov functions we shall extend these results by also employing a type of piecewise linear Lyapunov function. The analysis reveals that depending on properties of the constituent systems, either the quadratic Lyapunov function or the piecewise linear Lyapunov function is more suitable to establish stability of the switched system.

The chapter is organised as follows. In the following section the piecewise linear Lyapunov function under consideration is introduced and elementary properties for its existence for LTI systems derived. In Section 3.3 available pencil conditions for the existence of common quadratic Lyapunov Lyapunov functions are summarised along with some technical lemmas that shall proof useful for deriving the main results. In Section 3.4 the main stability result is proven and the existence of the two types of common Lyapunov function for different system-classes is discussed.

3.2 The unic Lyapunov function

The following example illustrates that for certain switched systems common quadratic Lyapunov functions might not be the best choice to establish asymptotic stability.

Example 3.1 *Consider the switch system* (3.1) *with constituent system matrices*

$$A_1 = \begin{pmatrix} -4.3 & -4.6 \\ -0.6 & -1.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1.8 & -2.4 \\ 9.3 & -11.9 \end{pmatrix}. \tag{3.2}$$

The eigenvalues of A_1 are approximately -5 and -0.4; A_2 has the eigenvalues -10.0 and -0.1. Hence, both LTI systems are stable and have a quadratic Lyapunov function each. A sample trajectory of the switched system (3.1) *with system matrices* (3.2) *is depicted in Figure 3.1. Here switching occurs every $\frac{\pi}{5}$ time-units; the initial state is $x(0) = (1,1)^{\mathsf{T}}$. The switching signal was chosen to be little stabilising to demonstrate some 'worst' behaviour of the system.*

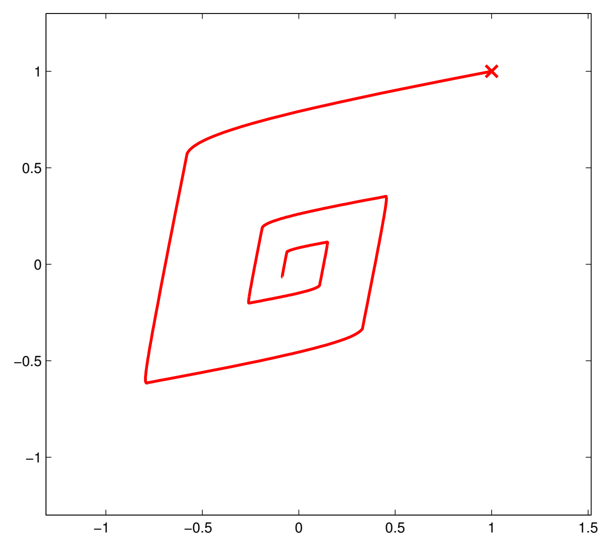


Figure 3.1: Sample trajectory of the switched system in Example 3.2

The switching instances are clearly visible as corners of the trajectory in the phase-plane. For such switched system a common quadratic Lyapunov function does not appear to be the optimal choice, since its level-sets are ellipses in the state-space and therefore are less suitable to account for such corners in the trajectories. (In fact, we shall see later that there exists no CQLF for the constituent systems in this example.) The shape of the trajectory suggests to look for a type of Lyapunov function that is not continuously differentiable and therefore can account for the characteristics of the trajectories. In this section we propose a piecewise linear Lyapunov function for the analysis of switched systems and derive properties for its existence for LTI systems.

3.2.1 Definition

In this section we introduce a piecewise linear Lyapunov function defined on the basis of the l_1-norm. It can be considered as a special case of the polyhedral Lyapunov functions considered in [Pol97] and of the norm-based Lyapunov functions discussed in [KAS92].

Definition 3.1 (Unic Lyapunov function candidate) *Let $L_d \in \mathbb{R}^{2\times 2}$ be a non-singular diagonal matrix, and $T \in \mathbb{R}^{2\times 2}$, nonsingular, then a unic Lyapunov function candidate is defined as*

$$V(x) \quad = \quad \left\| L_d T^{-1} x \right\|_1 . \tag{3.3}$$

$V(x)$ has a unique minimum at $x = 0$, is positive definite otherwise and radially unbounded and therefore satisfies the conditions for a Lyapunov function candidate in Theorem 2.8.

The level-sets given by $\left\{x_c \,\middle|\, V(x_c) = c,\ c > 0\right\}$ describe parallelograms in the phase-plane with vertices along the vectors given by the columns of T^{-1} (c.f. Figure 3.2).

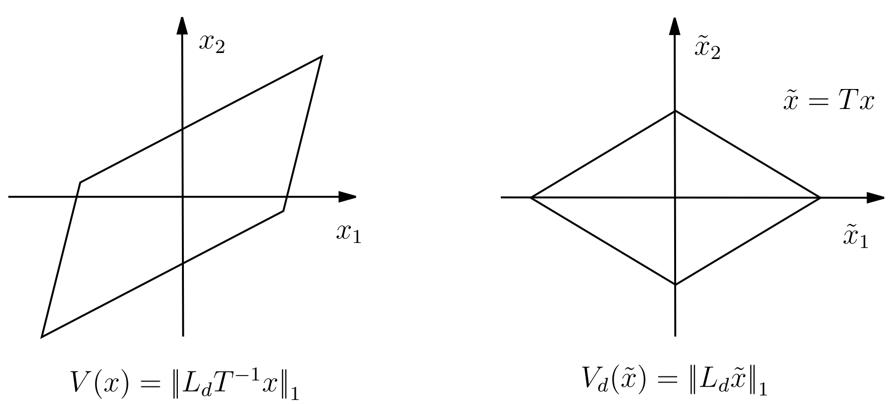


Figure 3.2: Level-curves of the unic and diagonal unic Lyapunov function

For the special case where T is the identity I the vertices of the level-sets of $V(x)$ are on the co-ordinate axis. We refer to such Lyapunov functions $V_d(x) = \|L_d x\|$ as diagonal unic Lyapunov function. Note, that any LTI system Σ_A for which a unic Lyapunov function $V(x)$ exists, the co-ordinate transformation $\tilde{x} = T^{-1}x$ yields the system $\Sigma_{\tilde{A}}$, with $\tilde{A} = T^{-1}AT$ such that $V_d(\tilde{x}) = \|L_d\tilde{x}\|$ is a diagonal unic Lyapunov function for $\Sigma_{\tilde{A}}$.

The stability conditions presented in this chapter are based on eigenvalue properties of the system matrices. Since the eigenvalues of A are invariant under the similarity transformation T we shall derive most results in terms of the existence of a diagonal unic Lyapunov function for $\Sigma_{\tilde{A}}$. If we can find a similarity transformation T such that $V_d(\tilde{x})$ is a diagonal unic Lyapunov function for $\Sigma_{\tilde{A}}$ then $V(x) = \left\|L_d T^{-1}x\right\|$ is a unic Lyapunov function for Σ_A. Hence, there exists a unic Lyapunov function $V(x)$ for Σ_A if and only if there exists a diagonal unic Lyapunov function $V_d(x)$ for $\Sigma_{\tilde{A}}$.

Before analysing the existence of a common unic Lyapunov function for a set of LTI systems, we shall examine the existence of a diagonal unic Lyapunov function for a single LTI system.

Diagonal unic Lyapunov functions for LTI systems

In the following we consider the existence of a diagonal unic Lyapunov function (dULF) for a single LTI system Σ_A. We shall use simple geometric relations of the entries of the system matrix A to construct a dULF and derive conditions for which such Lyapunov function exists. These conditions can be confirmed by application of Theorem 2.22 for norm-based Lyapunov functions.

Consider the LTI system with system matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

Since the dULF has its vertices on the co-ordinate axis of the phase-plane we consider the flow Ax of the system along the co-ordinate axis as depicted in Figure 3.3. We construct a dULF by choosing the level-set along the flow at the point $(-1, 0)$. The flow of Σ_A is given by the vector $\dot{x} = [-a \ -c]^\mathsf{T}$. It follows that the Lyapunov surface crosses the x_2-axis at the point $(0, -\frac{c}{a})$. By symmetry the other vertices of $V(x)$ are given by $(1, 0)$ and $(0, \frac{c}{a})$.

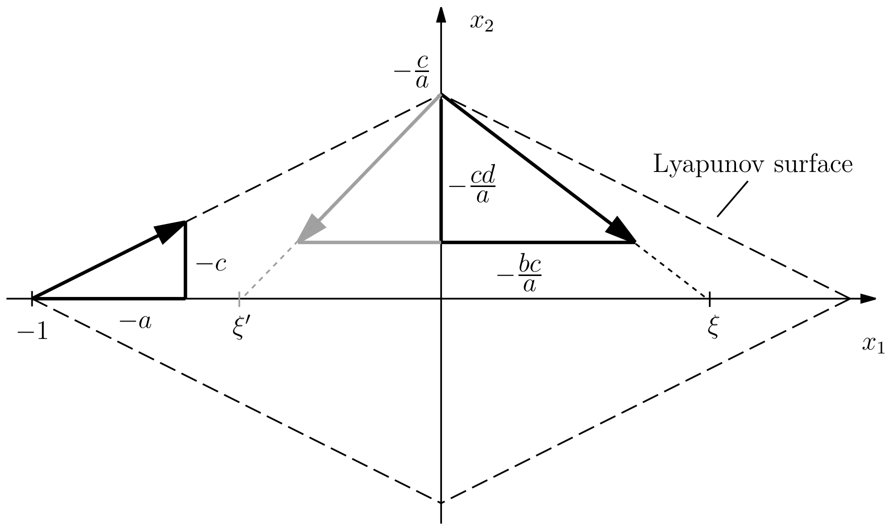


Figure 3.3: Construction of a diagonal unic Lyapunov function for an LTI system Σ_A.

By the construction of the Lyapunov function we can deduce conditions on the entries of A. For the point $(-1, 0)$ being a vertex of the Lyapunov function we require that $a < 0$ and by symmetry $d < 0$. Next, let ξ denote the projection along the flow at $(0, -\frac{c}{a})$ onto the x_1-axis. The flow at the point $(0, -\frac{c}{a})$ has to cut inside the Lyapunov surface (at the limit being parallel). Thus we require $|\xi| < 1$. Basic geometry reveals

$$\begin{aligned} |\xi| &= \left|\frac{bc}{ad}\right| < 1 \\ |bc| &< |ad|. \end{aligned}$$

The flow along the faces of the parallelogram is a convex combination of the flow at the vertices. Therefore the above conditions are necessary and sufficient. These observations are summarised in the following lemma.

Lemma 3.2 *A necessary and sufficient condition for the system Σ_A with system matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}$$

to have a diagonal unic Lyapunov function $V_d(x)$ is that

$$a^2 d^2 > b^2 c^2 \tag{3.4}$$

$$\textit{and} \qquad a, d < 0. \tag{3.5}$$

Proof. Since the dULF is a piecewise linear Lyapunov function we can apply Theorem 2.22 using the version for l_1-norms that can be found in [KAS92]. $V(x) = \|L_d x\|_1$ is a Lyapunov function for the system Σ_A if and only if

$$\max_{k\in\{1,2\}} q_{kk} + \sum_{\substack{l=1 \\ l\neq k}}^{2} |\, q_{lk} \,| \quad < \quad 0 \tag{3.6}$$

where

$$\begin{aligned} L_d A &= Q L_d \\ \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \\ \Leftrightarrow \qquad Q &= \begin{pmatrix} a & \frac{l_1}{l_2}\, b \\ \frac{l_2}{l_1}\, c & d \end{pmatrix}. \end{aligned}$$

The inequality (3.6) for the diagonal dominance of Q yields then

$$\max\left\{ a + \left| \frac{l_2}{l_1}\, c \right|,\ d + \left| \frac{l_1}{l_2}\, b \right| \right\} \quad < \quad 0$$

It follows immediately that $a, d < 0$. Further

$$\begin{aligned} a + \left| \frac{l_2}{l_1}\, c \right| < 0 \quad &\Leftrightarrow \quad \left| \frac{l_2}{l_1} \right| < \frac{-a}{|\, c\,|} \\ d + \left| \frac{l_1}{l_2}\, b \right| < 0 \quad &\Leftrightarrow \quad \left| \frac{l_2}{l_1} \right| > \frac{|\, b\,|}{-d} \end{aligned}$$

Hence,

$$\left| \frac{b}{d} \right| < \left| \frac{l_2}{l_1} \right| < \left| \frac{a}{c} \right| \tag{3.7}$$

□

Corollary 3.3 (dULF for LTI systems) *Let Σ_A be an LTI system that satisfies the condition for the existence of a dULF in Lemma 3.2. Then $V_d(x) = \|L_d x\|_1$ is a Lyapunov function for Σ_A with*

$$L_d \;=\; \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \qquad \textit{with} \qquad \left| \frac{b}{d} \right| < \left| \frac{l_2}{l_1} \right| < \left| \frac{a}{c} \right|, \quad l_1, l_2 \in \mathbb{R}. \tag{3.8}$$

Proof. This follows immediately from inequality (3.7) of the proof of Lemma 3.2. □

3.2.2 Eigenvalue condition for LTI systems

The above necessary and sufficient conditions for the existence of a diagonal unic Lyapunov function are based on the entries of the system matrix and therefore are co-ordinate dependent. We shall now derive a condition on the eigenvalues of the systems matrix A that guarantees the existence of a unic Lyapunov function $V_d(x)$ for the system Σ_A.

A necessary and sufficient condition for the existence of a quadratic Lyapunov function for an LTI system Σ_A is that the eigenvalues of A lie in the open left half of the complex plane. Analogous, we can identify a region in the complex plane that contains the eigenvalues of A if and only if there exists a unic Lyapunov function for Σ_A.

Definition 3.4 (45°-Region) *The* 45°*-Region is the open subset* Ω_{45} *of the complex plane defined by*

$$\Omega_{45} = \left\{ s \;\middle|\; \operatorname{Re}\{s\} < -\left|\operatorname{Im}\{s\}\right| \right\}_{\mathbb{C}} \tag{3.9}$$

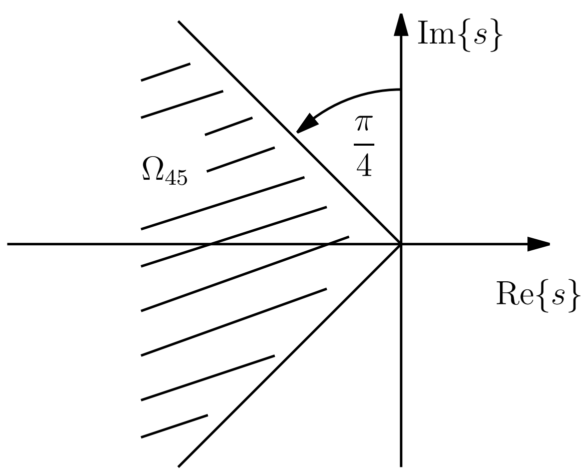


Figure 3.4: The 45°-Region in the complex plane.

Lemma 3.5 *The eigenvalues of the matrix* $A \in \mathbb{R}^{2\times 2}$ *lie within the* 45°*-Region if and only if A is Hurwitz and*

$$\operatorname{tr}^2 A \quad > \quad 2\det A$$

where $\operatorname{tr}^2 A$ *denotes* $(trace(A))^2$.

Proof. The eigenvalues of A are given by

$$\lambda \quad = \quad \frac{1}{2}\mathrm{tr}A \pm \sqrt{\frac{1}{4}\mathrm{tr}^2 A - \det A}\,.$$

If the eigenvalues are real, the Hurwitz condition suffices.

Otherwise, the eigenvalues are complex if and only if

$$\mathrm{tr}^2 A - 4\det A \quad < \quad 0\,. \tag{3.10}$$

The eigenvalues λ_i are in the 45°-Region if and only if $\mathrm{Re}\{\lambda\} < 0$ and

$$\begin{aligned}
\left|\mathrm{Re}\{\lambda\}\right| &> \left|\mathrm{Im}\{\lambda\}\right| \\
\left|\frac{\mathrm{tr}A}{2}\right| &> \left|\sqrt{\frac{1}{4}\mathrm{tr}^2 A - \det A}\right| \\
\frac{\mathrm{tr}^2 A}{4} &> \left|\frac{1}{4}\mathrm{tr}^2 A - \det A\right| \\
\frac{\mathrm{tr}^2 A}{4} &> -\frac{1}{4}\mathrm{tr}^2 A + \det A \\
\mathrm{tr}^2 A &> 2\det A
\end{aligned}$$

□

Based on the previous findings we can derive a necessary and sufficient condition for the existence of a unic Lyapunov function by constraining the eigenvalue location of A to the 45°-Region.

Theorem 3.6 (Unic Lyapunov Function for LTI systems) *Let A be a Hurwitz matrix in $A \in \mathbb{R}^{2\times 2}$. There exists a unic Lyapunov function for the LTI system Σ_A if and only if the eigenvalues of A lie within the 45°-Region.*

Proof. We prove sufficiency and necessity of the theorem separately.
Sufficiency: Consider the LTI system Σ_A where $A \in \mathbb{R}^{2\times 2}$ is Hurwitz with complex eigenvalues $\lambda_{1/2} = \mu \pm i\nu$ with $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$ and eigenvectors $x \pm iy$ with $x, y \in \mathbb{R}^{2\times 1}$. Then the eigenvalue equation $A(x \pm iy) = (\mu \pm i\nu)(x \pm iy)$ yields

$$\begin{aligned}
Ax &= \mu x - \nu y \qquad \text{and} \\
Ay &= \nu x + \mu y\,.
\end{aligned}$$

The vectors x, y cannot be linearly dependent as we can see from the following contradiction. If x, y are linearly dependent, i.e. $y = kx, k \in \mathbb{R} \setminus \{0\}$, we get

$$\begin{aligned} Ax &= (\mu - \nu k)x \\ Akx &= (\nu + \mu k)x \\ \Rightarrow \quad k(\mu - \nu k) &= \nu + \mu k \\ 0 &= (1 + k^2)\nu\,. \end{aligned}$$

This equality does not hold if the eigenvalues of A are non-real ($\nu \neq 0$). Thus x, y have are linearly independent.

Consider the non-singular matrix $T = \begin{pmatrix} x & y \end{pmatrix} \in \mathbb{R}^{2\times 2}$. Similarity transformation yields

$$\begin{aligned} \tilde{A} &= T^{-1}AT \\ &= \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}. \end{aligned}$$

It follows by Lemma 3.2 that there exists a diagonal unic Lyapunov function $V_d(x) = \|L_d x\|_1$ for $\Sigma_{\tilde{A}}$ since $|\,\mu\,| > |\,\nu\,|$ and $\mu < 0$. Hence, $V(x) = \left\|L_d T^{-1} x\right\|_1$ is a unic Lyapunov function for Σ_A. The invariance of the eigenvalues under similarity transformation proves the condition.

For the case that A has negative real eigenvalues, a simple modal transformation, i.e. diagonalisation of A, reveals the existence of a diagonal unic Lyapunov function.

Necessity: For necessity we show by contradiction that there is no similarity transformation such that $\Sigma_{\tilde{A}}$ has a diagonal unic Lyapunov function if the eigenvalues of A lie outside the 45°-Region.

Suppose we have an appropriately transformed matrix

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with eigenvalues outside the 45°-Region. Applying Lemma 3.5 yields

$$\begin{aligned} \mathrm{tr}^2 A &\leq 2 \det A \\ (a+d)^2 &\leq 2ad - 2bc \\ a^2 + d^2 &\leq -2bc. \end{aligned}$$

Since $a^2 + d^2 \geq 0$, both sides are non-negative and we get after squaring

$$\left(a^2 + d^2\right)^2 \leq 4b^2c^2. \tag{3.11}$$

Assume that there exists a diagonal unic Lyapunov function for $\Sigma_{\tilde{A}}$. Then Lemma 3.2 provides

$$b^2c^2 \quad < \quad a^2d^2.$$

With inequality (3.11) we get

$$\left(a^2+d^2\right)^2 \quad \leq \quad 4b^2c^2 \quad < \quad 4a^2d^2.$$

But this implies

$$\begin{aligned}\left(a^2+d^2\right)^2 &\quad < \quad 4a^2d^2 \\ \left(a^2-d^2\right)^2 &\quad < \quad 0.\end{aligned}$$

Since the left side of the above inequality is non-negative, the inequality does not hold and thus the system $\Sigma_{\tilde{A}}$ cannot have a diagonal unic Lyapunov function. This implies that there exists no unic Lyapunov function for Σ_A.

□

3.3 Preliminary results and definitions

3.3.1 Matrix Pencil

The stability condtion derived in this chapter is formulated in terms of the eigenvalues of the convex combination $\alpha A_1 + (1-\alpha)A_2$. In accordance with the convention of earlier publications in this area we define the matrix-pencil as follow.

Definition 3.7 (Matrix Pencil) *The matrix pencil $\sigma_\alpha[A_1, A_2]$ is the convex combination of the matrices A_1, A_2, given by*

$$\sigma_\alpha[A_1, A_2] = \alpha A_1 + (1-\alpha)A_2, \quad \alpha \in [0,1]. \tag{3.12}$$

Note, that the eigenvalues of a matrix depend continuously on its entries [HJ85]. Therefore, the eigenvalues $\sigma_\alpha[A_1, A_2]$ describe a continuous locus in the complex plane while α varies in $[0,1]$. We shall refer to that locus as the *eigenvalue locus* of the matrix pencil.

Recall from Section 2.4 that $V(x)$ is a Lyapunov function for Σ_{A_1} and Σ_{A_2} if and only if it is also a Lyapunov for all LTI systems $\Sigma_{\bar{A}}$ with $\bar{A} = \alpha A_1 + (1-\alpha)A_2$ for $\alpha \in [0,1]$ (Lemma 2.12).

As we shall see, the eigenvalues of the matrix pencil can provide insight into the stability of the switched system. For instance, consider the case were the matrix pencil $\sigma_\alpha[A_1, A_2]$ has an eigenvalue with non-negative real part for some $\alpha \in [0, 1]$. Thus, for some $\alpha_0 \in [0, 1]$, we obtain the unstable system matrix $A_0 = \alpha_0 A_1 + (1 - \alpha_0) A_2$ for which no Lyapunov function exists. It follows immediately from Lemma 2.12 that there exists no common Lyapunov function of any type for Σ_{A_1} and Σ_{A_2} and hence the switched system (3.1) is not asymptotically stable for arbitrary switching. Therefore, a necessary condition for asymptotic stability of the switched system (3.1) is that the eigenvalues of the matrix pencil lie in the open left half-plane for all $\alpha \in [0, 1]$. We shall say, such matrix pencil is Hurwitz. For details on the instability of switched systems see [SOCC00].

For the existence of a common *quadratic* Lyapunov function for two second order LTI systems we recall the following two theorems which shall be applied for deriving the main results in this chapter.

Theorem 3.8 (Sufficient condition for CQLF) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices with real eigenvalues. A sufficient condition for the existence of a CQLF for the LTI systems Σ_{A_1} and Σ_{A_2} is that the matrix pencil $\sigma_\alpha[A_1, A_2]$ has negative, real eigenvalues for all $\alpha \in [0, 1]$.*

Proof: see [SN97].[1]

Theorem 3.9 (Necessary and sufficient condition for CQLF) *Let A_1 and A_2 be Hurwitz matrices in $\mathbb{R}^{2\times 2}$. A necessary and sufficient condition for the existence of a CQLF for the LTI systems Σ_{A_1} and Σ_{A_2} is that the pencils $\sigma_\alpha[A_1, A_2]$ and $\sigma_\alpha[A_1, A_2^{-1}]$ have eigenvalues in the open left half of the complex plane for all $\alpha \in [0, 1]$.*

Proof: see [SN99].

The following lemma provides a simple condition for testing the Hurwitz property of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ for second order systems [WS01], [SN02]. This makes some calculations in this chapter more tractable and, more importantly, simplifies the condition for the CQLF existence to a simple eigenvalue-test.

[1] Originally, this theorem was restricted to matrices with distinct eigenvalues. However, it can be shown that the theorem extends to matrices with identical eigenvalues, by introducing ε as the difference between the eigenvalues and letting $\varepsilon \to 0$.

Lemma 3.10 (Matrix-product equivalence) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices. The matrix-pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0,1]$ if and only if the matrix product $A_2^{-1}A_1$ has no negative, real eigenvalue.*

Proof. According to the assumptions A_1 and A_2 are both Hurwitz. Hence,

$$\mathrm{tr}\big(\sigma_\alpha[A_1, A_2]\big) \;=\; \alpha\,\mathrm{tr}A_1 + (1-\alpha)\,\mathrm{tr}A_2 \;<\; 0, \qquad \text{for all} \quad \alpha \in [0,1].$$

It follows that $\sigma_\alpha[A_1, A_2]$ cannot have purely imaginary eigenvalues. Thus, the matrix pencil $\sigma_\alpha[A_1, A_2]$ is non-Hurwitz if and only if there exists an $\alpha_0 \in (0,1)$ for which $\sigma_{\alpha_0}[A_1, A_2]$ is singular. Then,

$$\begin{aligned}
\Big(\alpha_0 A_1 + (1-\alpha_0)A_2\Big)v &= 0\\
\Big(A_2^{-1}A_1 + \frac{1-\alpha_0}{\alpha_0}\,I_n\Big)v &= 0\\
A_2^{-1}A_1\,v &= -\frac{1-\alpha_0}{\alpha_0}\,v\,.
\end{aligned}$$

Hence, $A_2^{-1}A_1$ has a negative, real eigenvalue $-\frac{1-\alpha_0}{\alpha_0}$ if and only if the matrix pencil $\sigma_\alpha[A_1, A_2]$ is non-Hurwitz. □

Lemma 3.10 provides simple algebraic conditions for the Hurwitz property of the matrix pencil which will be used extensively in the proofs of the main results.

Moreover, it allows the reformulation of Theorem 3.9 as an eigenvalue condition. Note that the eigenvalue loci of $\sigma_\alpha[A_1, A_2]$ and $\sigma_\alpha[A_2, A_1]$ are identical. Hence, $A_2^{-1}A_1$ has real negative eigenvalues if and only if $A_1^{-1}A_2$ does. Moreover, by symmetry and Theorem 3.9 we can formulate the following corollary.

Corollary 3.11 *Given two Hurwitz matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$. A necessary and sufficient condition for the existence of a CQLF for the LTI systems Σ_{A_1} and Σ_{A_2} is that the matrix products $A_1^{-1}A_2$ and A_1A_2 have no negative, real eigenvalues.*

3.3.2 Some technicalities

Before we commence deriving the main results in this chapter, we collect some properties of the matrix pencil that are frequently used in the later proofs.

Even though the theorems in this chapter will be stated in terms of the matrix pencil $\sigma_\alpha[A_1, A_2]$, for algebraic simplicity the proofs will be given in terms of the related matrix pencil $\sigma_\gamma[A_1, A_2]$ defined by

$$\sigma_\gamma[A_1, A_2] \quad = \quad A_1 + \gamma A_2\,, \qquad \gamma \in [0, \infty). \tag{3.13}$$

The eigenvalue loci of the matrix pencils $\sigma_\alpha[A_1, A_2]$ and $\sigma_\gamma[A_1, A_2]$ have equal properties in terms of their relative location to sectors in the complex plane with vertex at the origin and the negative real axis as bisector.

Lemma 3.12 *The matrix-pencil $\sigma_\alpha[A_1, A_2]$ with $A_1, A_2 \in \mathbb{R}^{n\times n}$ has an eigenvalue that is real, complex or lies in the 45°-Region, if and only if the eigenvalues of A_1 or $\sigma_\gamma[A_1, A_2]$ for $\gamma \geq 0$ has an eigenvalue with the same property, respectively.*

Proof. Consider the matrix-pencil $\sigma_\alpha[A_1, A_2]$ for $\alpha \in (0, 1]$. Then,

$$\sigma_\alpha[A_1, A_2] \quad = \quad \alpha\left(A_1 + \frac{1-\alpha}{\alpha}A_2\right)\,, \qquad \forall\, \alpha \neq 0.$$

Define $f : (0, 1] \to [0, \infty)$ with $\gamma = f(\alpha) = \frac{1-\alpha}{\alpha}$. Then,

$$\sigma_\alpha[A_1, A_2] \quad = \quad \frac{1}{\gamma+1}\,(A_1 + \gamma A_2) \quad = \quad \frac{1}{\gamma+1}\,\sigma_\gamma[A_1, A_2] \qquad \gamma \geq 0\,.$$

The positive scaling factor $\frac{1}{\gamma+1}$ does not effect the considered properties of the eigenvalues (real, complex or within the 45°-Region) and can therefore be omitted. The singularities $\gamma = -1$ and $\alpha = 0$ are not in the considered intervals.

For $\alpha = 0$ the eigenvalues of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ equal the eigenvalues of A_1. This completes the proof. □

The following lemma describes the dependency of $\det \sigma_\gamma[A_1, A_2]$ on γ. For ease of exposition we shall use the following notation throughout:

$$\begin{aligned} \mathrm{tr}_i \quad &\equiv \quad \mathrm{tr} A_i \\ \Delta_i \quad &\equiv \quad \det A_i \qquad \forall\, i \in \mathcal{I}. \end{aligned}$$

Lemma 3.13 *Let A_1, A_2 be nonsingular matrices in $\mathbb{R}^{2\times 2}$, then*

$$\det\bigl(\sigma_\gamma[A_1, A_2]\bigr) \quad = \quad \Delta_2\gamma^2 + \Delta_1 \mathrm{tr}\left(A_1^{-1}A_2\right)\gamma + \Delta_1\,. \tag{3.14}$$

Proof. Let A_1 and A_2 be given by

$$A_1 \quad = \quad \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad A_2 \quad = \quad \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

then the determinant of the matrix pencil is given by

$$\begin{aligned}
\det(A_1+\gamma A_2) &= (a_1+\gamma b_1)(a_4+\gamma b_4)-(a_2+\gamma b_2)(a_3+\gamma b_3)\\
&= (b_1b_4-b_2b_3)\gamma^2+(a_4b_1+a_1b_4-a_3b_2-a_2b_3)\gamma+(a_1a_4-a_2a_3)\\
&= \Delta_2\gamma^2+(a_4b_1+a_1b_4-a_3b_2-a_2b_3)\gamma+\Delta_1
\end{aligned}$$

Consider

$$\Delta_1 A_1^{-1}A_2 = \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_4b_1-a_2b_3 & a_4b_2-a_2b_4 \\ a_1b_3-a_3b_1 & a_1b_4-a_3b_2 \end{pmatrix}$$

With $\Delta_1\mathrm{tr}\big(A_1^{-1}A_2\big)=\mathrm{tr}\big(\Delta_1A_1^{-1}A_2\big)$ the result follows. □

Similar elementary calculations reveal the following relation between the traces of the matrix-products A_1A_2 and $A_1^{-1}A_2$.

Lemma 3.14 *Let A_1, A_2 be nonsingular matrices in $\mathbb{R}^{2\times2}$, then*

$$\mathrm{tr}(A_1A_2) = \mathrm{tr}_1\mathrm{tr}_2-\Delta_1\mathrm{tr}\left(A_1^{-1}A_2\right) \tag{3.15}$$

Proof. This relation can be shown using similar elementary calculations as in the proof of Lemma 3.13 above and is therefore omitted. □

3.4 Sufficient condition for stability

In this section a number of theorems are presented that relate the eigenvalue locus of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ and the 45°-Region to the existence of a common unic or common quadratic Lyapunov function. The aim is to derive a sufficient condition for asymptotic stability of (3.1). Since the existence of the two types of Lyapunov function under consideration differs for various subclasses we shall discuss the following cases separately:

(i) the system matrices $A_1, A_2 \in \mathbb{R}^{2\times2}$ are Hurwitz and have real eigenvalues;

(ii) the system matrices $A_1, A_2 \in \mathbb{R}^{2\times2}$ are Hurwitz and have complex eigenvalues;

(iii) the system matrices $A_1, A_2 \in \mathbb{R}^{2\times2}$ are Hurwitz one of which having real eigenvalues and the other having complex eigenvalues.

Combining the results obtained in the discussion of these cases yields to a compact and readily applicable condition for asymptotic stability of the switched system (3.1) for arbitrary switching signals.

Theorem 3.15 (45° Criterion) *The switched linear system* (3.1) *with* $\mathcal{A}=\{A_1, A_2\}$*,* $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *and arbitrary switching signals complying with Definition 2.1 is asymptotically stable if the eigenvalue-locus of the matrix pencil* $\sigma_\alpha[A_1, A_2]$*,* $\alpha \in [0, 1]$ *lies completely in the* 45°*-Region.*

This theorem is proven in the course of this section and is a direct consequence of the Theorems 3.8, 3.16, 3.21 and 3.23.

Even though the above stability condition holds for all three cases, the analysis of the different system classes gives insight into their different characteristics. The existence of the type of Lyapunov function differs for each system class which gives insight into different properties of the system. By discussing each case separately, we attempt to highlight those differences.

3.4.1 Constituent systems with real eigenvalues

We begin our discussion by switched systems consisting of subsystems with real eigenvalues, i.e. $\mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{2\times 2}$ where A_1, A_2 are Hurwitz and have real eigenvalues. By Theorem 3.8 there exists a common quadratic Lyapunov function (CQLF) for Σ_{A_1} and Σ_{A_2} if the eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ is Hurwitz and real for all $\alpha \in [0, 1]$. However the eigenvalues of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ can be complex for some $\alpha \in [0, 1]$ even if the eigenvalues of A_1, A_2 are real. The subsystems of the introductory Example 3.1 have such property as is shown by the eigenvalue locus depicted in Figure 3.5.

For such cases we need to resort to the eigenvalue locus of $\sigma_\alpha[A_1, A_2^{-1}]$ to establish quadratic stability (c.f. Theorem 3.9). Unfortunately, there is not always a CQLF for such systems. Using Lemma 3.10 we can quickly verify that the matrix pencil $\sigma_\alpha[A_1, A_2]$ of Example 3.1 is non-Hurwitz since the product $A_1^{-1}A_2$ has negative real eigenvalues. Thus the LTI systems Σ_{A_1} and Σ_{A_2} have no CQLF. However we shall show that asymptotic stability of the switched system (3.1) can still be established via the existence of a common unic Lyapunov function, given that the eigenvalue locus lies in the 45°-Region.

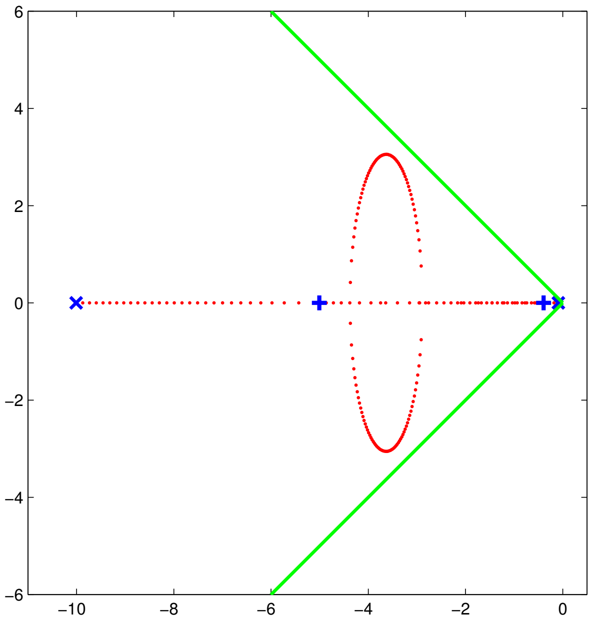


Figure 3.5: Eigenvalue locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ of the switched system in Example 3.1. The eigenvalues of A_1 and A_2 are indicated by crosses, the dots represent the eigenvalues of $\sigma_\alpha[A_1, A_2]$ for some values of $\alpha \in [0, 1]$.

Theorem 3.16 (Sufficient condition for common unic LF) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices with real eigenvalues. The linear systems Σ_{A_1} and Σ_{A_2} have a common unic Lyapunov function if the matrix pencil $\sigma_\alpha[A_1, A_2]$ has complex eigenvalues for some $\alpha \in (0, 1)$ and its eigenvalue-locus stays within the $45°$-Region for all $\alpha \in [0, 1]$.*

The key-idea of the proof is to find points in the phase-plane where the flows of Σ_{A_1} and Σ_{A_2} are co-linear. We apply a similarity transformation defined by these points of common flow and establish the existence of a common diagonal unic Lyapunov function (dULF) for the transformed systems $\Sigma_{\tilde{A}_1}$ and $\Sigma_{\tilde{A}_2}$ under the conditions of the theorem. The existence of a common dULF for the transformed systems implies the existence of a common unic Lyapunov function for the original systems. Before presenting the proof of Theorem 3.16, we discuss some preliminary findings.

Lemma 3.17 *Given two matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ with negative, real eigenvalues for which the matrix pencil $\sigma_\alpha[A_1, A_2]$ has complex eigenvalues for some $\alpha \in [0, 1]$. Then the matrix $S = A_2^{-1}A_1$ has distinct, positive, real eigenvalues.*

Proof. As noted in Section 3.3.2, the matrix pencil $\sigma_\alpha[A_1, A_2]$ has complex eigenvalues for $\alpha \in [0,1]$ if and only if $\sigma_\gamma[A_1, A_2]$ does so for $\gamma \geq 0$. Hence, there exist some $\gamma > 0$ for which

$$\mathrm{tr}^2\big(\sigma_\gamma[A_1, A_2]\big) \quad < \quad 4\det\big(\sigma_\gamma[A_1, A_2]\big)\,.$$

On both sides of the inequality we have second order polynomials in γ given by

$$p_1(\gamma) \quad = \quad \mathrm{tr}^2\sigma_\gamma \quad = \quad (\,\mathrm{tr}_1 + \gamma\,\mathrm{tr}_2\,)^2$$

and, using the expression (3.14) of Lemma 3.13 we obtain

$$p_2(\gamma) \quad = \quad 4\det\sigma_\gamma \quad = \quad 4\Delta_2\,\gamma^2 + 4\,\Delta_1\mathrm{tr}\left(A_1^{-1}A_2\right)\,\gamma + 4\Delta_1\,.$$

Since A_1 and A_2 are Hurwitz it follows that their traces are negative. Hence, $p_1(\gamma)$ has a negative double zero at $\gamma_0 = -\frac{\mathrm{tr}_1}{\mathrm{tr}_2}$ and is positive otherwise.

Since the determinant of A_2 is positive, the second parabola $p_2(\gamma)$ is positive for $\gamma \to \pm\infty$. Further the eigenvalues of σ_γ are real for $\gamma = 0$ and $\gamma \to \infty$. Hence, $p_1(\gamma) > p_2(\gamma)$ for $\gamma = 0$ and $\gamma \to \infty$. Moreover, for the values $\gamma_\mathrm{c} > 0$ where the matrix-pencil σ_γ has complex eigenvalues, we have $p_1(\gamma_\mathrm{c}) < p_2(\gamma_\mathrm{c})$. It follows that there are two values $\gamma_{\mathrm{c}1}, \gamma_{\mathrm{c}2} > 0$ for which $p_1(\gamma) = p_2(\gamma)$. These observations are shown graphically in Figure 3.6.

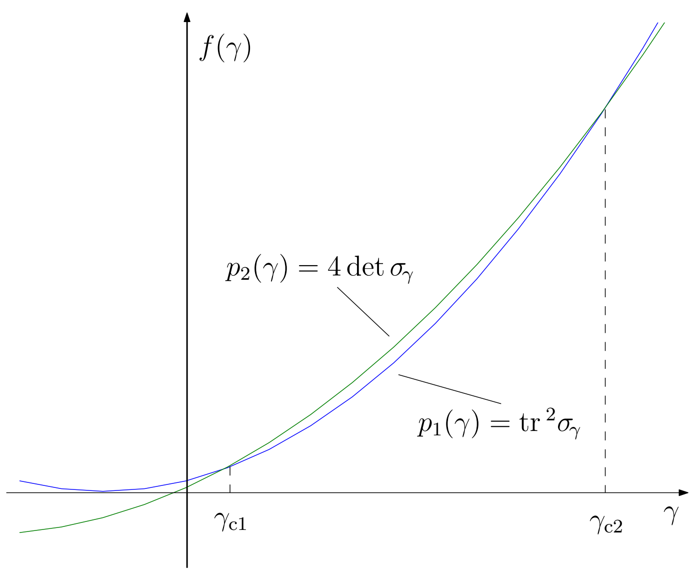


Figure 3.6: Geometric relation of parabolas $p_1(\gamma)$ and $p_2(\gamma)$

Two parabolas can intersect at most twice, i.e. $p_2(\gamma) < p_1(\gamma)$ for $\gamma < 0$. Since

$p_1(\gamma)$ has a negative zero, $p_2(\gamma)$ has to have two real zeros, one of which is less than $\gamma_0 = -\frac{\mathrm{tr}_1}{\mathrm{tr}_2} < 0$. With those two findings we can prove the result:

(i) $p_2(\gamma)$ has two real zeros:

$$\begin{aligned} \Delta_1^2 \mathrm{tr}^2\left(A_1^{-1}A_2\right) - 4\Delta_1\Delta_2 &> 0 \\ \mathrm{tr}^2\left(A_1^{-1}A_2\right) &> 4\,\frac{\Delta_2}{\Delta_1} \\ \mathrm{tr}^2 S &> 4\det S > 0\,. \end{aligned}$$

From the first inequality follows that $S = A_1^{-1}A_2$ has distinct eigenvalues since $\frac{1}{4}\mathrm{tr}^2 S - \det S \neq 0$. The last inequality implies that S has real eigenvalues with the same sign.

(ii) $p_2(\gamma)$ has a negative zero:

$$\begin{aligned} -\Delta_1 \mathrm{tr}\left(A_1^{-1}A_2\right) &< 0 \\ \mathrm{tr} S &> 0 \end{aligned}$$

Hence, S has a positive eigenvalue and with the above argument the proof is complete. □

The above lemma implies that the matrices A_1 and A_2 have points in the phase-plane where the flow is co-linear, i.e.

$$\begin{aligned} A_2\, v_{\mathrm{cf},i} &= k_i A_1 v_{\mathrm{cf},i} \\ A_1^{-1}A_2\, v_{\mathrm{cf},i} &= k_i v_{\mathrm{cf},i}\,, \qquad\qquad k_i > 0. \end{aligned}$$

The vectors of common flow $v_{\mathrm{cf},i}$, $i = 1, 2$ are the eigenvectors of $S = A_1^{-1}A_2$ and $k_i > 0$ are the eigenvalues of S. Since the eigenvalues k_i are distinct we can construct the non-singular transformation matrix

$$T = \begin{pmatrix} v_{\mathrm{cf},1} & v_{\mathrm{cf},2} \end{pmatrix} \tag{3.16}$$

consisting of the eigenvectors of S. Then the transformed system matrices $\tilde{A}_i = T^{-1}A_iT$ have co-linear flow along the x_1- and x_2-axes. These observations are summarised in the following corollary.

Corollary 3.18 *Given two matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ such that $A_1^{-1}A_2$ has real, positive eigenvalues k_1, k_2. Then there exists a non-singular transformation T such that*

$$\tilde{A}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tilde{A}_2 = \begin{pmatrix} k_1 a & k_2 b \\ k_1 c & k_2 d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ and $k_1, k_2 > 0$.

Proof. Let D_S be the diagonal matrix with eigenvalues k_1, k_2 of S such that $D_S = T^{-1}ST$. Then

$$\begin{aligned}\tilde{A}_2 &= T^{-1}A_2T \\ &= T^{-1}A_1A_1^{-1}A_2T \\ &= \tilde{A}_1T^{-1}ST \\ &= \tilde{A}_1D_S\end{aligned}$$

□

Note, if V_d is a diagonal unic Lyapunov function for $\Sigma_{\tilde{A}_1}$, then V_d is also a Lyapunov function for $\Sigma_{\tilde{A}_2}$. This follows immediately from condition (3.8).

With that we can end the preliminary discussion and proof Theorem 3.16.

Proof. [Proof of Theorem 3.16]

By assumption, the matrix pencil $\sigma_\alpha[A_1, A_2]$ has complex eigenvalues for some $\alpha \in (0, 1)$. Therefore, we can apply Corollary 3.18 and consider the matrix pencil of the transformed matrices $\tilde{A}_1, \tilde{A}_2$. The eigenvalues are invariant under the similarity transformation T, hence the eigenvalue loci of $\sigma_\alpha[\tilde{A}_1, \tilde{A}_2]$ and $\sigma_\alpha[A_1, A_2]$ are identical. Applying Lemma 3.12 the eigenvalues of the matrix pencil $\sigma_\alpha[\tilde{A}_1, \tilde{A}_2]$, $\alpha \in [0, 1]$ lie in the same sectors as the eigenvalues of $\sigma_\gamma[\tilde{A}_1, \tilde{A}_2]$, $\gamma \geq 0$. We consider

$$\begin{aligned}\sigma_\gamma[\tilde{A}_1, \tilde{A}_2] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \gamma \begin{pmatrix} k_1a & k_2b \\ k_1c & k_2d \end{pmatrix} \\ &= \begin{pmatrix} \left[1 + k_1\gamma\right]a & \left[1 + k_2\gamma\right]b \\ \left[1 + k_1\gamma\right]c & \left[1 + k_2\gamma\right]d \end{pmatrix} \\ &= \left[1 + k_1\gamma\right] \begin{pmatrix} a & \frac{1+k_2\gamma}{1+k_1\gamma}b \\ c & \frac{1+k_2\gamma}{1+k_1\gamma}d \end{pmatrix}.\end{aligned}$$

The eigenvalues the matrix $A \in \mathbb{R}^{n\times n}$ lie in the 45°-Region if and only if the eigenvalues of $\left[1 + k_1\gamma\right]A$ do. Hence, the positive factor $(1 + k_1\gamma)$ does not change the eigenvalue-locus of $\sigma_\gamma[A_1, A_2]$ with respect to the 45°-Region. Therefore we shall omit that factor in the further discussion.

Define the parameter

$$\tilde{\gamma} \quad = \quad \frac{1 + k_2\gamma}{1 + k_1\gamma}\,.$$

Without loss of generality we can assume that $k_2 > k_1$. Then $\tilde{\gamma} \in \Gamma = \left[1, \frac{k_2}{k_1}\right)$ for $\gamma \geq 0$. We obtain

$$\sigma_{\tilde{\gamma}} \quad = \quad \begin{pmatrix} a & \tilde{\gamma} b \\ c & \tilde{\gamma} d \end{pmatrix} \qquad \tilde{\gamma} \in \Gamma\,.$$

The existence of a diagonal unic Lyapunov function can be established by extracting the relation of the trace and determinant of $\sigma_{\tilde{\gamma}}$:

$$\begin{aligned} \mathrm{tr}^2 \sigma_{\tilde{\gamma}} &= (a + \tilde{\gamma} d)^2 \\ \det \sigma_{\tilde{\gamma}} &= (ad - bc)\tilde{\gamma}\,. \end{aligned}$$

By our assumptions we get the following relations:

(i) the matrices A_1, A_2 have real eigenvalues, i.e. at the endpoints of Γ we have

$$\mathrm{tr}^2 \sigma_{\tilde{\gamma}} \quad > \quad 4 \det \sigma_{\tilde{\gamma}}\,;$$

(ii) the matrix pencil has complex eigenvalues for some $\tilde{\gamma}$ in the interior of Γ

$$\mathrm{tr}^2 \sigma_{\tilde{\gamma}} \quad < \quad 4 \det \sigma_{\tilde{\gamma}}\,;$$

(iii) the eigenvalues of the matrix pencil are within the 45°-Region for all $\tilde{\gamma} \in \Gamma$

$$\mathrm{tr}^2 \sigma_{\tilde{\gamma}} \quad > \quad 2 \det \sigma_{\tilde{\gamma}}\,.$$

These relations are illustrated in Figure 3.7. From (i) and (ii) we get that the parabola $\mathrm{tr}^2 \sigma_{\tilde{\gamma}}$ crosses the straight line given by $4 \det \sigma_{\tilde{\gamma}}$ twice in Γ.

Note that $\mathrm{tr}^2 \sigma_{\tilde{\gamma}}$ has a unique minimum and a double zero at $\tilde{\gamma}_0 = -\frac{a}{d}$. The above argument implies that $\tilde{\gamma}_0 < 0$, hence a and d have the same sign. Since $\tilde{A}_1$ is Hurwitz, $\mathrm{tr} \tilde{A}_1 < 0$ and it follows that $a, d < 0$.

Relation (iii) gives that $\mathrm{tr}^2 \sigma_{\tilde{\gamma}}$ does not intersect with $2 \det \sigma_{\tilde{\gamma}}$ for $\tilde{\gamma} \in \Gamma$. Since the minimum of the parabola is zero, we can infer that there is no intersection for any $\tilde{\gamma} \in \mathbb{R}$. Hence,

$$d^2 \tilde{\gamma}^2 + 2bc\tilde{\gamma} + a^2 \quad > \quad 0 \qquad \forall\, \tilde{\gamma} \in \mathbb{R}\,.$$

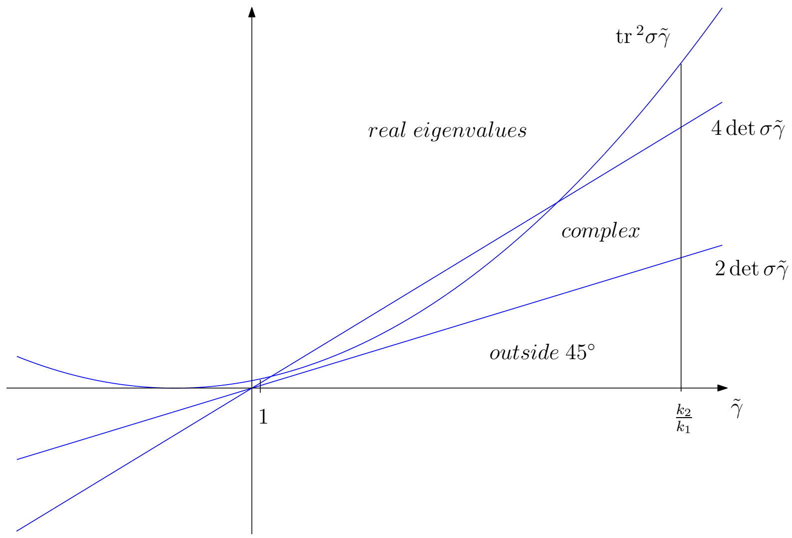


Figure 3.7: Geometrical relation of the trace and determinant of $\sigma_{\tilde{\gamma}}$.

That immediately yields the remaining condition (3.4) for the existence of a diagonal unic Lyapunov function

$$b^2c^2 - a^2d^2 \quad < \quad 0\,.$$

Thus, there exists a diagonal unic Lyapunov function for $\tilde{A}_1$.

It follows from Condition (3.8) that any diagonal unic Lyapunov function $V_d(x) = \|L_d x\|_1$ for $\Sigma_{\tilde{A}_1}$ is also a Lyapunov function for $\Sigma_{\tilde{A}_2}$. By similarity $V(x) = \left\|L_d T^{-1} x\right\|_1$ is a common unic Lyapunov function for Σ_{A_1} and Σ_{A_2}. □

Theorem 3.16 complements Theorem 3.8 to a sufficient condition for asymptotic stability of the switched system (3.1) with real eigenvalues. If the matrix pencil $\sigma_\alpha\,[A_1, A_2]$ has real eigenvalues for all $\alpha \in [0,1]$, Theorem 3.8 guarantees the existence of a common quadratic Lyapunov function. In case that the matrix pencil has complex eigenvalues for some $\alpha \in (0,1)$ a common unic Lyapunov function exists.

Corollary 3.19 (45° Criterion) *The switched linear system* (3.1) *with system matrices* $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *with real eigenvalues, is asymptotically stable for arbitrary switching signals if the eigenvalue-locus of the matrix pencil* $\sigma_\alpha\,[A_1, A_2]\,,\ \alpha \in [0,1]$ *lies completely in the* 45°*-Region.*

Discussion and application of the result

In the following we demonstrate the application of the result and discuss further implications of the 45° Criterion for the existence of the two types of Lyapunov function considered. In particular we are interested in how conservative the 45° Criterion is.

The following example demonstrates the application of the result and shows that the 45° Criterion extends upon quadratic stability. If is further demonstrated that an explicit unic Lyapunov function is immediately obtained by the application of the results in this chapter. This can prove valuable in practice when a stable switched system is to be designed.

Example 3.2 (No common quadratic LF, but common unic LF)
Consider the switched system in the introductory Example 3.1 with

$$A_1 = \begin{pmatrix} -4.3 & -4.6 \\ -0.6 & -1.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1.8 & -2.4 \\ 9.3 & -11.9 \end{pmatrix}.$$

Figure 3.5 shows that the eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ lies completely in the 45°-Region. However, the eigenvalues of the matrix product A_1A_2 are approximately $\{-35.94, -0.05\}$. Since they are both negative and real, by Lemma 3.10 no common quadratic Lyapunov function exists. But Theorem 3.16 guarantees the existence of a common unic Lyapunov function.

A specific unic Lyapunov function is obtained as follows. The matrix product $A_1^{-1}A_2$ has positive real eigenvalues such Σ_{A_1} and Σ_{A_2} have points of co-linear flow in the phase-plane. For the transformation matrix T given in (3.16) we obtain

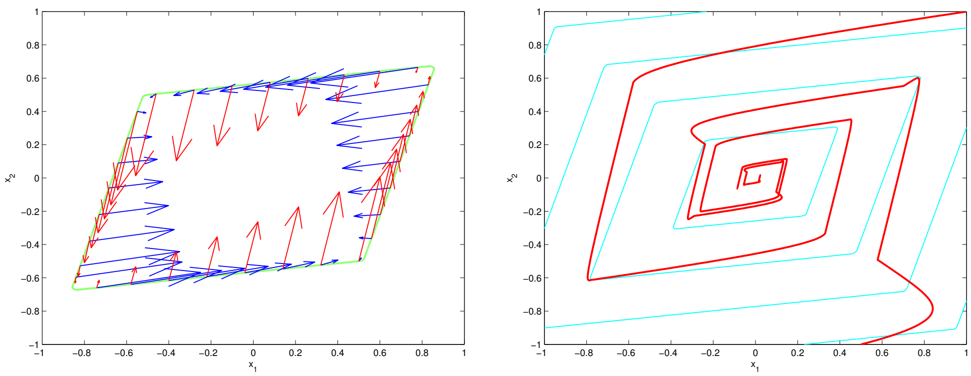
$$T = \begin{pmatrix} -0.7875 & 0.7235 \\ -0.6164 & -0.6903 \end{pmatrix}$$

yielding the transformed system matrices

$$\tilde{A}_1 = \begin{pmatrix} -5.1813 & -0.2827 \\ 2.9597 & -0.2187 \end{pmatrix} \quad \text{and} \quad \tilde{A}_2 = \begin{pmatrix} -0.0515 & -12.9902 \\ 0.0294 & -10.0485 \end{pmatrix}.$$

With Corollary 3.3 we obtain a common diagonal unic Lyapunov function for $\Sigma_{\tilde{A}_1}$ and $\Sigma_{\tilde{A}_2}$ by choosing L_d with $\frac{l_2}{l_1} \in [\,1.2927, 1.7506\,]$, where the interval is simply given by the ratio of the column entries of $\tilde{A}_1$. Common unic Lyapunov functions for Σ_{A_1} and Σ_{A_2} are immediately given by $V(x) = \left\|L_d T^{-1} x\right\|_1$.

A sample level-set of the unic Lyapunov function with $\frac{l_2}{l_1} = 1.52$ and the flow of the constituent systems are depicted in Figure 3.8a. Some sample trajectories of the switched system together with some level-sets of the Lyapunov function are shown in Figure 3.8b. □



(a) (b)

Figure 3.8: Unic Lyapunov function for the switched system in Example 3.2. Part (a) shows the flow of the vector fields of the constituent LTI systems on a level set of the unic Lyapunov $V(x) = \left\|L_d T^{-1} x\right\|_1$. Part (b) shows two sample trajectories of the switched system for the initial conditions $(1, 1)$ and $(0.5, -1)$, respectively, and some level sets of the Lyapunov function.

The above example demonstrates the need for the two types of Lyapunov function to formulate the stability condition in Corollary 3.19. Moreover, we can identify the existence of each type of Lyapunov function with the respective behaviour of the eigenvalue locus of $\sigma_\alpha[A_1, A_2]$. This suffices for stability of the switched system, but is by no means necessary for stability. However, the 45°-Region provides a necessary condition for the existence of both types of Lyapunov function, as the following theorem reveals.

Theorem 3.20 (Necessary Condition) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices with real eigenvalues and for which the matrix pencil $\sigma_\alpha[A_1, A_2]$ has eigenvalues outside the 45°-Region for some $\alpha \in [0, 1]$. Then*

(i) there exists no common unic Lyapunov function for Σ_{A_1} and Σ_{A_2};

(ii) there exists no common quadratic Lyapunov function for Σ_{A_1} and Σ_{A_2}.

Proof. Statement *(i)* follows directly from the preliminary discussion. If V is a common Lyapunov function for Σ_{A_1} and Σ_{A_2} then it is also a Lyapunov function for all systems defined by convex combinations $\alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0, 1]$, (Lemma 2.12).

However, if for some $\alpha_0 \in [0,1]$ the matrix pencil $\sigma_\alpha[A_1, A_2]$ has eigenvalues outside the 45°-Region, then there exists no unic Lyapunov function for the LTI system defined by $\alpha_0 A_1 + (1-\alpha_0)A_2$. Hence, Σ_{A_1} and Σ_{A_2} cannot share a common unic Lyapunov function.

To prove statement *(ii)* we show that under the given assumptions, the matrix product $A_1 A_2$ has real negative eigenvalues. Then by Corollary 3.11 there exists no CQLF.

If the matrix pencil has eigenvalues outside the 45°-Region then there exists a $\gamma_0 > 0$ for which

$$\mathrm{tr}^2 \sigma_\gamma \quad < \quad 2 \det \sigma_\gamma \,.$$

With (3.14) we get

$$(\mathrm{tr}_1 + \mathrm{tr}_2\, \gamma_0)^2 \quad < \quad 2\Delta_2 \gamma_0^2 + 2\Delta_1 \mathrm{tr}\left(A_1^{-1} A_2\right) \gamma_0 + 2\Delta_1$$

rearranging and substituting (3.15) yields

$$\left(\mathrm{tr}_2^2 - 2\Delta_2\right) \gamma_0^2 + 2\mathrm{tr}\big(A_1 A_2\big)\gamma_0 + \mathrm{tr}_1^2 - 2\Delta_1 \quad < \quad 0\,.$$

Since A_1 and A_2 have real eigenvalues, i.e. $\mathrm{tr}_i - 4\Delta_i > 0$, the above parabola is positive for $\gamma = 0$ and $\gamma \to \infty$. It follows that for the inequality to hold, the parabola has two positive real zeros. That implies $-\mathrm{tr}(A_1 A_2) > 0$ and

$$\begin{aligned} \mathrm{tr}^2\big(A_1 A_2\big) - \left(\mathrm{tr}_2^2 - 2\Delta_2\right)\left(\mathrm{tr}_1^2 - 2\Delta_1\right) \quad &> \quad 0 \\ \mathrm{tr}^2\big(A_1 A_2\big) - \mathrm{tr}_1^2 \mathrm{tr}_2^2 + 2\Delta_1 \mathrm{tr}_2^2 + 2\Delta_2 \mathrm{tr}_1^2 - 4\Delta_1\Delta_2 \quad &> \quad 0\,. \end{aligned}$$

Rearranging gives

$$\mathrm{tr}^2\big(A_1 A_2\big) - 4\Delta_1\Delta_2 \quad > \quad \frac{1}{2}\left(\mathrm{tr}_1^2 - 4\Delta_1\right)\mathrm{tr}_2^2 + \frac{1}{2}\left(\mathrm{tr}_2^2 - 4\Delta_2\right)\mathrm{tr}_1^2\,.$$

Since A_1, A_2 have real eigenvalues the righthand side of the inequality is positive. Hence, the product $A_1 A_2$ has real eigenvalues. Since its determinant is positive and its trace is negative, both eigenvalues are real and negative. This violates the necessary condition for CQLF existence in Corollary 3.11. □

The above theorem establishes that the 45°-Region is also a necessary condition for the existence of the two types of Lyapunov functions under consideration when A_1, A_2 have real eigenvalues. Although none of these Lyapunov functions exists when the eigenvalue-locus of $\sigma_\alpha[A_1, A_2]$ lies outside the 45°-Region for some α, the switched system might still be asymptotically stable. However, the following example demonstrates that the 45°-Region is the largest sector (with vertex at the origin and the negative real axis as bisector) that results in a sufficient stability condition of the kind in Theorem 3.15.

Example 3.3 *We can construct a limit system by choosing an eigenvalue of zero for both subsystems. Further, let the eigenvectors associated with these eigenvalues be* $(-1 \;\; 1)^\mathsf{T}$ *and* $(1 \;\; 1)^\mathsf{T}$*. For the remaining eigenvectors we choose* $(1 \;\; 0)^\mathsf{T}$ *and* $(0 \;\; 1)^\mathsf{T}$*, respectively, and let the associated eigenvalues be -100 and -200. The resulting system matrices are*

$$\tilde{A}_1 = \begin{pmatrix} -100 & -100 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{A}_2 = \begin{pmatrix} 0 & 0 \\ 200 & -200 \end{pmatrix} \tag{3.17}$$

The eigenvalue locus of such system just touches the boundary of the 45°-Region as shown in Figure 3.9. When switching slow enough, the trajectories describe a limit cycle as in Figure 3.10.

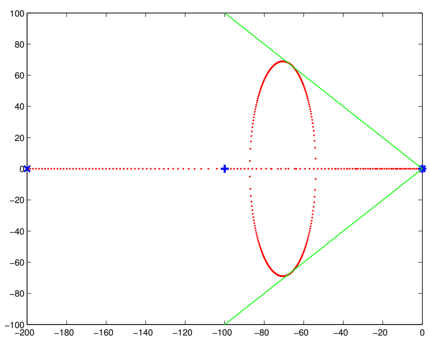


Figure 3.9: Eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ for the system in (3.17).

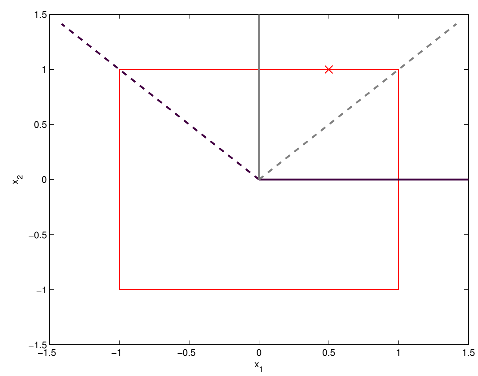


Figure 3.10: Sample trajectory and eigenvectors of the system in (3.17).

Small perturbations of such system can lead to switched system with unstable trajectories. Choose

$$\tilde{A}_1 = \begin{pmatrix} -100.00 & -99.99 \\ 0 & -0.01 \end{pmatrix} \quad \text{and} \quad \tilde{A}_2 = \begin{pmatrix} -0.02 & 0 \\ 202.01 & -200.00 \end{pmatrix}$$

The transition matrix $\Phi(\frac{40}{\pi}, 0) = e^{A_2 \frac{40}{\pi}} e^{A_1 \frac{40}{\pi}}$ *has a spectral radius of* 1.005*. Hence, the switched system is just unstable for switching signals with switching instances at every* $\frac{40}{\pi}$ *time-units. Of course, the eigenvalue locus is just outside the 45°-Region for some* $\alpha \in [0, 1]$*.* □

Summarising the results of this section, we can relate the different behaviour of the eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ to the existence of the two types of Lyapunov functions for second order systems with real eigenvalues. If the eigenvalue locus remains real and Hurwitz for all $\alpha \in [0, 1]$, then a common quadratic Lyapunov function exists (Theorem 3.8); if there are complex eigenvalues of $\sigma_\alpha[A_1, A_2]$ for some values $\alpha \in [0, 1]$, but the eigenvalue locus stays within the 45°-Region for all $\alpha \in [0, 1]$, then the existence of a common unic Lyapunov function is guaranteed (Theorem 3.16); and if the matrix pencil has eigenvalues outside the 45°-Region for some $\alpha \in [0, 1]$, then neither a common quadratic Lyapunov function nor a common unic Lyapunov function exists (Theorem 3.20). However, recall that the non-existence of those two types of Lyapunov function does not necessarily imply the instability of the switched system.

The 45° Criterion in Corollary 3.19 provides a compact and readily applicable condition for stability of second-order systems with real eigenvalues. In the following two sections we extend this extended this result to systems with arbitrary eigenvalues in the 45°-Region and discuss its implications for the existence of common unic and common quadratic Lyapunov functions.

3.4.2 Constituent systems with complex eigenvalues

In this section we analyse to what extent the results on the 45° Criterion in the previous section generalise to switched systems where $A_1, A_2 \in \mathbb{R}^{2\times 2}$ have complex eigenvalues. We will see that the stability condition formulated in Corollary 3.19 also holds for this system class. However, necessity for the considered types of Lyapunov functions cannot be established.

Theorem 3.21 (Sufficient condition for CQLF) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices with non-real eigenvalues in the 45°-Region. The systems Σ_{A_1} and Σ_{A_2} have a CQLF if the eigenvalues of the matrix pencil $\sigma_\alpha[A_1, A_2]$ lie in the 45°-Region for all $\alpha \in [0, 1]$.*

Proof. We shall proof the theorem by employing the sufficient condition for CQLF existence in Theorem 3.8. By assumption the matrix pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz, thus it remains to be shown that $\sigma_\alpha[A_1^{-1}, A_2]$ is Hurwitz. By Lemma 3.10 this is equivalent to the matrix product $A_1 A_2$ having no negative real eigenvalues.

By assumption and Lemma 3.12, the eigenvalue locus of the matrix-pencil $A_1 + \gamma A_2$ lies in the 45°-Region for all $\gamma \geq 0$. Hence,

$$\mathrm{tr}^2\,(A_1 + \gamma A_2) \quad > \quad 2\det\,(A_1 + \gamma A_2) \qquad \forall\,\gamma \geq 0\,.$$

Substituting the expressions (3.14) and (3.15) and rearranging yields

$$p(\gamma) \;=\; \left(\mathrm{tr}_2^2 - 2\Delta_2\right)\gamma^2 + 2\mathrm{tr}\,(A_1A_2)\,\gamma + \mathrm{tr}_1^2 - 2\Delta_1 \;>\; 0 \qquad \forall\,\gamma \geq 0\,. \tag{3.18}$$

By assumption, A_2 has complex eigenvalues within the 45°-Region and therefore $\mathrm{tr}_2^2 - 2\Delta_2 > 0$. Hence, the parabola $p(\gamma)$ has a minimum. Thus, inequality (3.18) holds if and only if (a) all zeros of $p(\gamma)$ are negative, or (b) $p(\gamma)$ has no real zeros. For the first case (a), we require that

$$\begin{aligned} -2\mathrm{tr}(A_1A_2) \quad &< \quad 0 \\ \mathrm{tr}(A_1A_2) \quad &> \quad 0\,. \end{aligned}$$

This, together with $\det(A_1A_2) = \Delta_1\Delta_2 > 0$ implies that A_1A_2 has eigenvalues in the open right half-plane.

In the latter case (b), we require

$$\begin{aligned} \mathrm{tr}^2(A_1A_2) - \left(\mathrm{tr}_2^2 - 2\Delta_2\right)\left(\mathrm{tr}_1^2 - 2\Delta_1\right) \quad &< \quad 0 \\ \mathrm{tr}^2(A_1A_2) - \mathrm{tr}_1^2\mathrm{tr}_2^2 + 2\Delta_2\mathrm{tr}_1^2 + 2\Delta_1\mathrm{tr}_2^2 - 4\Delta_1\Delta_2 \quad &< \quad 0 \\ \mathrm{tr}^2(A_1A_2) - 4\Delta_1\Delta_2 - \frac{1}{2}\left(\mathrm{tr}_1^2 - 4\Delta_1\right)\mathrm{tr}_2^2 - \frac{1}{2}\left(\mathrm{tr}_2^2 - 4\Delta_2\right)\mathrm{tr}_1^2 \quad &< \quad 0\,. \end{aligned}$$

According to our assumptions A_1, A_2 have non-real eigenvalues, thus $\mathrm{tr}_i^2 - 4\Delta_i < 0$. This leaves

$$\mathrm{tr}^2(A_1A_2) - 4\Delta_1\Delta_2 \quad < \quad \frac{1}{2}\left(\mathrm{tr}_1^2 - 4\Delta_1\right)\mathrm{tr}_2^2 + \frac{1}{2}\left(\mathrm{tr}_2^2 - 4\Delta_2\right)\mathrm{tr}_1^2 \;\;<\;\; 0\,.$$

Hence, A_1A_2 has complex eigenvalues.

In both cases (a) and (b) the matrix product A_1A_2 has no negative real eigenvalues. Thus by Lemma 3.10 and Theorem 3.8 a common quadratic Lyapunov function for Σ_{A_1} and Σ_{A_2} exists. □

With the above theorem the stability statement of Corollary 3.19 can be extended to second order systems with complex eigenvalues.

Discussion of the result

In the previous section we found that the 45° Criterion is also a necessary condition for the existence of a CQLF and CULF if A_1, A_2 have real eigenvalues. In the following we shall discuss whether the 45°-Region is of similar significance if the eigenvalues of the constituent systems are complex.

We start considering two examples where the eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ is outside the 45°-Region for some $\alpha \in [0, 1]$.

Example 3.4 *Consider the switched system* (3.1) *with system matrices*

$$A_1 = \begin{pmatrix} -1.67 & -0.83 \\ 1.60 & 0.23 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.51 & -1.45 \\ 0.34 & -1.31 \end{pmatrix} \tag{3.19}$$

The eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ is Hurwitz and lies outside the 45°-Region for some $\alpha \in [0, 1]$ as shown in Figure 3.11. However, the eigenvalues of the matrix product A_1A_2 are approximately $\lambda_{1,2} = -1.0259 \pm 0.2085i$. Thus there exists a CQLF for Σ_{A_1} and Σ_{A_2}.

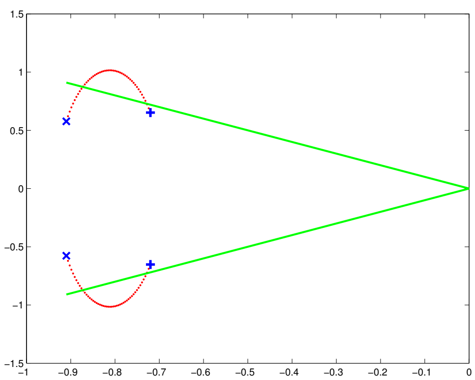


Figure 3.11: Eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ with (3.19).

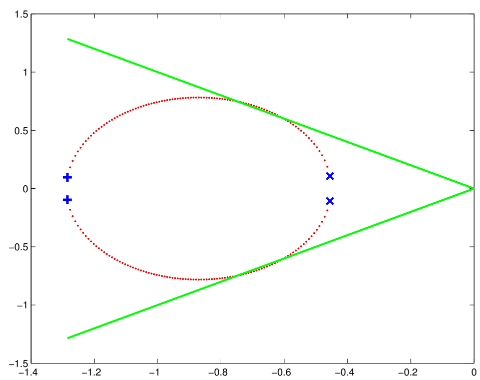


Figure 3.12: Eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ with (3.20).

Not surprisingly the 45° Criterion is not necessary for the existence of a CQLF for the case that the eigenvalues of the constituent systems are complex. However, it is not hard to find examples for which no CQLF exists when the eigenvalue locus leaves the 45°-Region for some $\alpha \in (0, 1)$.

In Figure 3.12 the eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ with

$$A_1 = \begin{pmatrix} -1.89 & -0.31 \\ 1.22 & -0.68 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0.32 & -1.08 \\ 0.56 & -1.23 \end{pmatrix} \tag{3.20}$$

is shown. The eigenvalues of the matrix product A_1A_2 are approximately $\lambda_1 = -0.8427$ and $\lambda_1 = -0.4169$. Hence, Σ_{A_1} and Σ_{A_2} do not have a common quadratic Lyapunov function. □

This observation suggests that it might be hard to find a larger sector that gives rise to a sufficient condition of the kind of Theorem 3.21 for the existence of a CQLF.

For systems with real eigenvalues two types of Lyapunov functions are required to prove stability for systems satisfying the 45° condition. In the current case with subsystems with complex eigenvalues only considering common quadratic Lyapunov functions was sufficient. It remains open whether the condition in Theorem 3.21 is also sufficient for the existence of a common unic Lyapunov function.

Example 3.5 Consider the switched system with

$$A_1 = \begin{pmatrix} -1.2 & 0.5 \\ -0.2 & -0.9 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -0.9 & -1.5 \\ 0.2 & -1.5 \end{pmatrix}.$$

The matrix product $S = A_1^{-1}A_2$ has the complex eigenvalues $0.9364 \pm 0.7221i$. Thus the systems Σ_{A_1} and Σ_{A_2} have no points of co-linear flow in the phase-plane. Therefore the method used in the previous section to construct a common unic Lyapunov function cannot be applied.

However, for this example we can find a diagonal unic Lyapunov function. Applying Corollary 3.3 yields that a dULF for system Σ_{A_1} satisfies

$$\left|\frac{b_1}{d_1}\right| < \left|\frac{l_2}{l_1}\right| < \left|\frac{a_1}{c_1}\right|$$
$$0.5556 < \frac{l_2}{l_1} < 6\,.$$

For system Σ_{A_2} we require $\frac{l_2}{l_1} \in [1, 4.5]$. Since the two intervals intersect we can choose $\frac{l_2}{l_1} = 2$ to obtain a common unic Lyapunov function $V(x) = \|L_d x\|$ with

$$L_d = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

satisfying both conditions. A level-set of $V(x)$ together with the flows of System Σ_{A_1} and Σ_{A_2} is shown in Figure 3.14. □

Certainly it is not always possible to find common diagonal unic Lyapunov function. Whether or not the 45° condition is sufficient for the existence of a common unic Lyapunov function for systems with complex eigenvalues has yet to be proven.

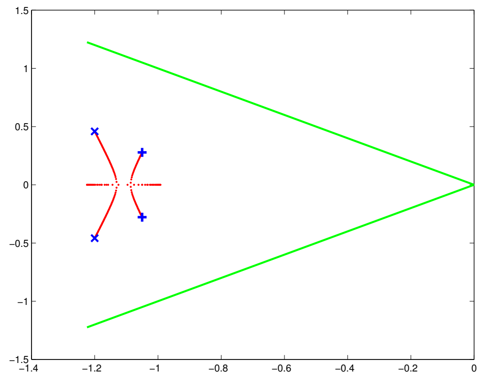


Figure 3.13: Eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ in Example 3.5.

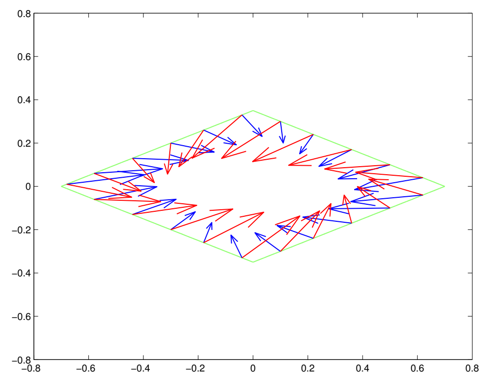


Figure 3.14: Common unic Lyapunov function for the system in Example 3.5.

3.4.3 The mixed case

After we have discussed the two cases of switched systems consisting of subsystems with real and complex eigenvalues, respectively, it remains to prove stability for the case where one matrix has real eigenvalues and the second matrix has complex eigenvalues in the 45°-Region.

Similar to the first case where both matrices A_1, A_2 have real eigenvalues, we will employ both types of Lyapunov function to establish stability. Before presenting the main result we shall note the following preliminary lemma.

Lemma 3.22 *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices, one of which has real eigenvalues and the other non-real eigenvalues. If the eigenvalues of the matrix pencil $\alpha A_1 + (1-\alpha)A_2$ lie in the 45°-Region for all $\alpha \in [0,1]$ then one of the following statements is true:*

(i) the matrices $A_1 A_2$ and $A_1^{-1}A_2$ have no real, negative eigenvalue;

(ii) the matrix $A_1^{-1}A_2$ has distinct, positive, real eigenvalues.

Proof. Without loss of generality we assume that the eigenvalues of A_1 are complex and the eigenvalues of A_2 are real. By assumption the matrix pencil $\sigma_\gamma[A_1, A_2]$ is Hurwitz. Thus the matrix product $A_1^{-1}A_2$ has no negative real eigenvalues (Lemma 3.10).

From the assumptions on the eigenvalues of A_1, A_2 and the matrix pencil $\sigma_\gamma[A_1, A_2]$ we can derive conditions on the relation of the trace of the matrix-pencil $\sigma_\gamma[A_1, A_2]$ to its determinant for $\gamma \geq 0$.

Consider the two parabolas given by

$$\begin{aligned} p_1(\gamma) &= \operatorname{tr}^2(A_1 + \gamma A_2) = (\operatorname{tr}_1 + \gamma \operatorname{tr}_2)^2 \\ p_2(\gamma) &= 2\det(A_1 + \gamma A_2) = 2\Delta_2\gamma^2 + 2\Delta_1 \operatorname{tr}\left(A_1^{-1}A_2\right)\gamma + 2\Delta_1 \,. \end{aligned}$$

The parabola $p_1(\gamma)$ has a unique negative zero at $\gamma_0 = -\frac{\operatorname{tr}_1}{\operatorname{tr}_2}$. Since the eigenvalues of the matrix pencil $\sigma_\gamma[A_1, A_2]$ lie in the 45°-Region for all $\gamma \geq 0$, it follows that $p_1(\gamma) > p_2(\gamma)$ for γ non-negative. The parabolas can only intersect for negative values of γ.

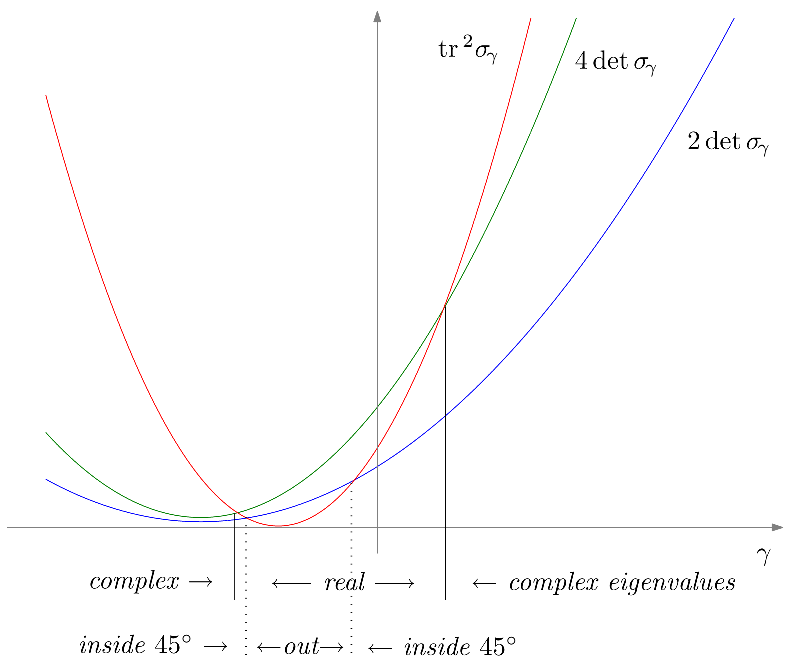


Figure 3.15: Geometric relation of the trace and determinant of the matrix pencil in Lemma 3.22.

Consider the case *(i)* where $p_1(\gamma)$ and $p(\gamma)$ have an intersection for $\gamma < 0$. Hence,

$$(\operatorname{tr}_1 + \gamma\operatorname{tr}_2)^2 - 2\Delta_2\gamma^2 - 2\Delta_1\operatorname{tr}\left(A_1^{-1}A_2\right)\gamma - 2\Delta_1 \quad = \quad 0\,.$$

With Lemma 3.14 we get

$$\left(\operatorname{tr}_2^2 - 2\Delta_2\right)\gamma^2 + 2\operatorname{tr}(A_1A_2)\gamma + \operatorname{tr}_1^2 - 2\Delta_1 \quad = \quad 0\,.$$

This polynomial in γ has a negative root only if $\operatorname{tr}(A_1A_2) > 0$. Since the determinant $\det(A_1A_2) = \Delta_1\Delta_2 > 0$ we conclude that the matrix product A_1A_2 has eigenvalues in the right half-plane.

In case *(ii)* the parabolas $p_1(\gamma)$ and $p_2(\gamma)$ do not intersect for $\gamma \in \mathbb{R}$. This implies that $p_2(\gamma)$ has two real roots, at least one of which is negative. Consider now the parabola $p_3(\gamma) = 2p_2(\gamma) = 4\det\sigma_\gamma$, which of course has also two real zeros. We obtain

$$\begin{aligned} \mathrm{tr}^2\left(A_1^{-1}A_2\right) - 4\frac{\Delta_2}{\Delta_1} &> 0 \\ \mathrm{tr}^2\left(A_1^{-1}A_2\right) &> 4\det\left(A_1^{-1}A_2\right) . \end{aligned}$$

Hence, the matrix product $A_1^{-1}A_2$ has distinct real eigenvalues.

By assumption A_1 has complex eigenvalues, thus $p_3(0) > p_1(0)$ while $p_1(0) > p_2(0)$ (because of the 45° Condition). Hence, the two real zeros of $p_2(\gamma)$ and $p_3(\gamma)$ are either both positive or both negative. Since we require that $p_2(\gamma)$ has a negative zero it follows that both zeros are negative. Therefore

$$\begin{aligned} -2\Delta_1\mathrm{tr}\left(A_1^{-1}A_2\right) &< 0 \\ \mathrm{tr}\left(A_1^{-1}A_2\right) &> 0\,. \end{aligned}$$

Since the determinant $\det\left(A_1^{-1}A_2\right) = \frac{\Delta_2}{\Delta_1} > 0$ it follows that the matrix product $A_1^{-1}A_2$ has positive real eigenvalues. □

We can now prove the stability result for the mixed case.

Theorem 3.23 (Sufficient condition for stability) *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices, one of which has real eigenvalues and the other non-real eigenvalues. The switched linear system* (3.1) *is asymptotically stable if the eigenvalues of the matrix pencil $\alpha A_1 + (1-\alpha)A_2$ lie in the 45°-Region for all $\alpha \in [0, 1]$.*

Proof. We will establish stability of the switched system by showing that under the given assumptions there either *(i)* exists a common quadratic or *(ii)* a common unic Lyapunov function for the LTI systems Σ_{A_1} and Σ_{A_1}.

Consider the two cases of Lemma 3.22. In case *(i)* it follows with Corollary 3.11 that there exists a CQLF for the systems Σ_{A_1} and Σ_{A_1}.

In the second case the matrix $S = A_1^{-1}A_2$ has distinct positive real eigenvalues. Hence we can transform the system matrices as in Corollary 3.18 such that the points of common flow are along the co-ordinate axis. We can establish the existence of a common diagonal unic Lyapunov function for the transformed systems by similar arguments as in the proof of Theorem 3.16 on page 62. □

In the following we present two examples that illustrate the results in this section.

Example 3.6 *Consider the switched system* (3.1) *with system matrices*

$$A_1 = \begin{pmatrix} -0.4 & -0.4 \\ 2.4 & -1.4 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -0.1 & 0 \\ 0.5 & -0.8 \end{pmatrix}$$

The eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ *is in the* 45°*-Region as shown in Figure 3.16. The eigenvalues of* $A_1^{-1}A_2$ *are complex. Thus there exist no points of co-linear flow. However, the eigenvalues of the matrix product* A_1A_2 *are* $\lambda_1 = 0.8154$ *and* $\lambda_2 = 0.1446$*. Hence, there exists a common quadratic Lyapunov function. A level set of the quadratic Lyapunov function* $V(x) = x^\mathsf{T}Px$ *with*

$$P = \begin{pmatrix} 2.3552 & 0.0582 \\ 0.0582 & 0.6033 \end{pmatrix}$$

is shown in Figure 3.17. □

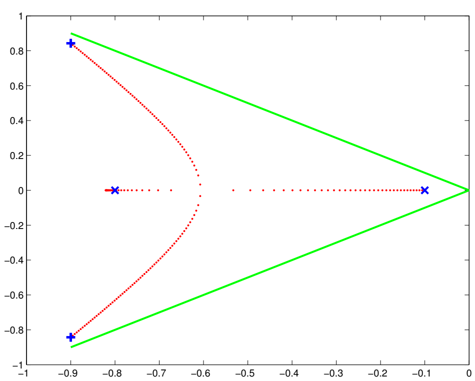


Figure 3.16: Eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ in Example 3.6.

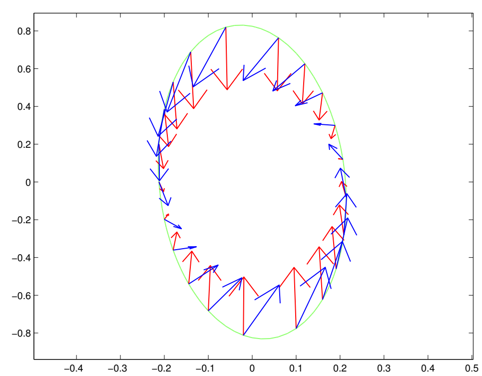


Figure 3.17: Common quadratic Lyapunov function for the system in Example 3.6.

Example 3.7 *This example demonstrates that the* 45° *Criterion extends stability results obtain by using only common quadratic Lyapunov functions. Consider the switched system* (3.1) *with system matrices*

$$A_1 = \begin{pmatrix} 0.3 & -0.5 \\ 1.3 & -1.9 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -1.2 & -1.2 \\ 0.4 & -0.6 \end{pmatrix}$$

The eigenvalues of the matrix product A_1A_2 *are approximately* $\lambda_1 = -0.87$ *and* $\lambda_2 = -0.11$*. Hence no CQLF exists for* Σ_{A_1} *and* Σ_{A_2}*. Yet, the eigenvalue locus of* $\sigma_\alpha[A_1, A_2]$ *lies in the* 45°*-Region as shown in Figure 3.18.*

We can construct a common unic Lyapunov function following the same procedure as in Example 3.5. A level set of the unic Lyapunov function $V(x) = \left\| L_d T^{-1} x \right\|_1$ with

$$L_d = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.3 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0.8253 & -0.6278 \\ 0.5647 & 0.7784 \end{pmatrix}$$

is shown in Figure 3.17. □

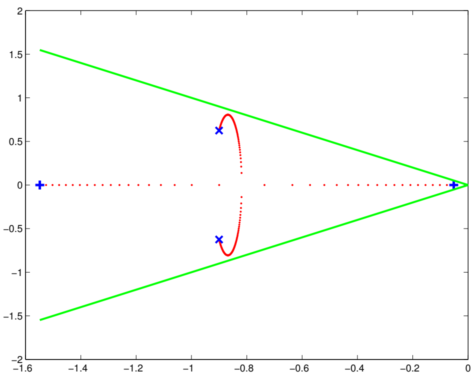


Figure 3.18: Eigenvalue-locus of the matrix pencil $\sigma_\alpha[A_1, A_2]$ in Example 3.7.

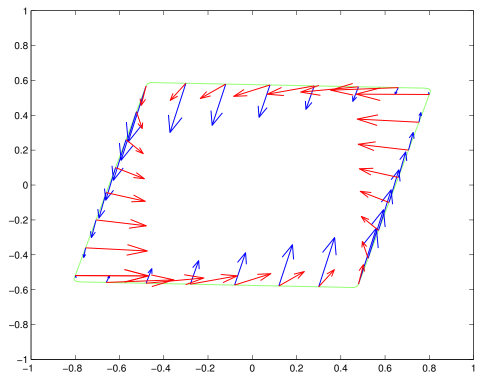


Figure 3.19: Common unic Lyapunov function for the system in Example 3.7.

3.5 Conclusions and outlook

The main contribution of this chapter is a sufficient condition for the asymptotic stability switched systems consisting of two second-order subsystems. The formulation of the condition in terms of the eigenvalues of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ ensures independence of co-ordinate transformations and is therefore readily applicable. In case of the existence of a common unic Lyapunov function, an explicit range of Lyapunov functions is directly obtained by the derivations. Therefore the stability of the switched system can be incorporated as a parameter constraint into the design process of the switched system.

Stability of the switched system is established by employing two different types of Lyapunov functions, quadratic and unic. It is shown that the stability condition can only be established after introduction of the latter type which accounts for certain characteristics of the considered switched systems.

The existence of either type of Lyapunov function has been analysed in relation to characteristics of the eigenvalue locus of the matrix pencil for different system classes.

We note that the unic Lyapunov function appears more suitable whenever one of the constituent systems has real eigenvalues. For this system class the existence of common quadratic Lyapunov functions can yield conservative results.

For switched systems with real eigenvalues it has been shown that the 45° Criterion is also a necessary condition for the existence of either type of Lyapunov function considered. In this context the 45°-Region is the largest region (with vertex at the origin and the negative real axis as bisector) for which eigenvalue locus conditions of the form of Theorem 3.15 can be obtained.

Future work

Unfortunately, it is not possible to generalise the 45° Criterion in Theorem 3.15 to switched systems with higher order subsystems.

Consider the switched system with system matrices

$$A_1 = \begin{pmatrix} -0.4 & 1.4 & 0 \\ 0.6 & -0.6 & -0.5 \\ 0.3 & 0.8 & -0.6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.9 & -0.5 & -1.0 \\ -1.0 & 0.4 & 0.8 \\ 1.1 & -1.0 & -1.7 \end{pmatrix}. \tag{3.21}$$

The eigenvalue locus of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ lies completely in the 45°-Region as shown in Figure 3.20. However, the spectral radius of the transition matrix

$$\Phi(2\pi, 0) = e^{A_2\pi} e^{A_1\pi}$$

is approximately 1.35. Hence, the switching signal with switching instances at every π time-units results in unstable behaviour [Rug96].

However, empirical studies suggest that the 45° Criterion might hold for switched linear systems where the system matrices of the constituent systems are given in the companion form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

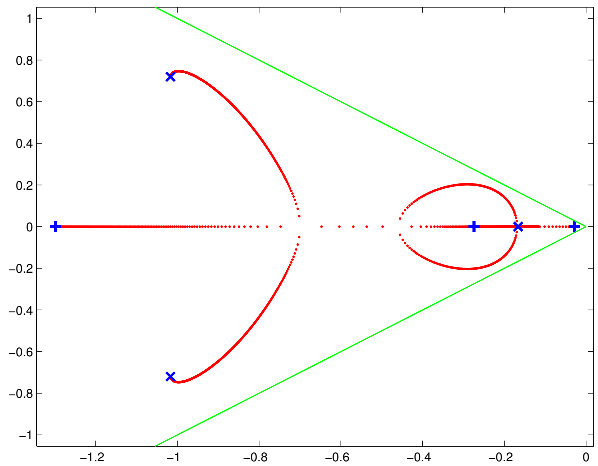


Figure 3.20: Eigenvalue locus of $\sigma_\alpha[A_1, A_2]$ for the system matrices (3.21).

Such system matrices are obtained for systems described by the time-varying differential equation

$$y^{(n)} = \sum_{k=0}^{n-1} a_k(t) y^{(k)} + u \tag{3.22}$$

where the coefficients $a_k(t)$ are piecewise constant and change discontinuously within finite parameter sets.

For the generalisation of our result two problems arise. Firstly it is important to determine a suitable type of Lyapunov function that can be employed to establish stability. Secondly, although the empirical studies suggest that the 45°-Region might yield sufficient conditions for the stability of this class of systems, a different region could be more significant for the considered system class. In this context the Theorems 2.23 and 2.24 on piecewise linear Lyapunov functions for LTI systems might prove valuable; both in determining the number of faces of the Lyapunov function needed and in indicating the appropriate spectral region.

Chapter 4

Spectral conditions for classical stability results

In this chapter we formally relate a class of switched linear systems to the non-linear system known as Lur'e system. It is shown that the switched system is asymptotically stable for arbitrary switching if and only if the Lur'e system is absolutely stable. This implication allows for the mutual application of stability results obtained in each area. We extend a result from [SN03a] to derive spectral versions of classical stability conditions for Lur'e systems. Some examples highlight the benefits of these new formulations.

4.1 Introductory remarks

In this chapter we consider the asymptotic stability under arbitrary switching signals of switched linear systems that consist of two subsystems of order n and whose system matrices $A_i \in \mathcal{A}$ are in companion form. In Section 2.4 it is shown that a common Lyapunov function for the constituent systems Σ_{A_1} and Σ_{A_2} is also a Lyapunov function for all LTI systems $\Sigma_{\bar{A}}$ with $\bar{A} = \alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$. We shall use this observation to establish the equivalence of the asymptotic stability of the system class defined above and the classical problem of absolute stability of single-input single-output Lur'e systems.

Given these observations, the question arises, how the stability results obtained for Lur'e systems relate to problems from the analysis of switched systems. In this chapter we establish that results from the stability theory of both system classes are mutually applicable, and further, that results from the Lur'e problem can give new insights into the stability properties of switched linear systems. More specifically, recently it has been shown in [SN03a] that the Circle Criterion derived for the SISO Lur'e system can be utilised to derive a necessary and sufficient condition for the existence of a CQLF for the switched system in terms of the product of the system matrices. The main contribution in this chapter is the generalisation of this result that makes it applicable to a wide range of stability conditions formulated in terms of the frequency response.

The chapter is organised as follows. In the first section the single-input single-output Lur'e system is introduced and the equivalence of the respective stability problems is shown. In the second section the main lemma is derived which relates the classical frequency response inequality to spectral properties of a matrix product. The frequency-response inequality in question appears in a large number of classical analysis tools. Therefore, the matrix product equivalence has potentially many implications in various areas. Some of these implications are discussed in Sections 4.4 and 4.6.

4.2 The Lur'e problem and switched linear systems

4.2.1 Lur'e systems and absolute stability

Lur'e systems are nonlinear feedback systems consisting of a linear time-invariant forward path and a memoryless (possibly time-varying) nonlinearity in the feedback path. The block structure of such system is shown in Figure 4.1.

The dynamics of the system are given by

$$\dot{x}(t) = Ax(t) + bu(t) \tag{4.1a}$$

$$y(t) = c^{\mathsf{T}}x(t) + du(t) \tag{4.1b}$$

$$z(t) = \phi\,(t, y(t)) \tag{4.1c}$$

$$u(t) = -z(t) \tag{4.1d}$$

where $x(t) \in \mathbb{R}^n, y(t), z(t), u(t), d \in \mathbb{R}$, the system matrix $A \in \mathbb{R}^{n \times n}$ and b, c are

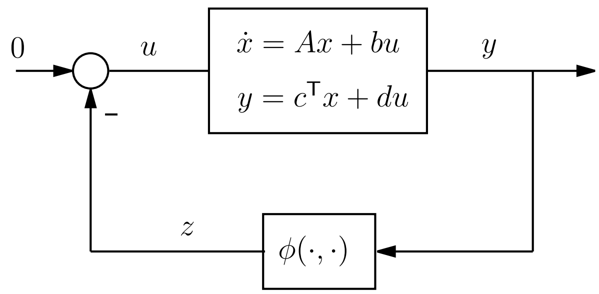


Figure 4.1: Block diagram of the single-input single-output Lur'e system.

vectors in $\mathbb{R}^n$. In accordance with convention the quadruplet $(A, b, c^{\mathsf{T}}, d)$ is assumed to be a minimal realisation of the proper rational transfer function

$$G(s) \quad = \quad c^{\mathsf{T}}\big(sI - A\big)^{-1}b + d \tag{4.2}$$

such that (A, b) is completely controllable and (A, c^{T}) is completely observable.[1]

The non-linearity $\phi(\cdot,\cdot)$ is a time-varying scalar function $\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying

$$k_1 y^2 \quad \leq \quad \phi(t, y)y \quad \leq \quad k_2 y^2 \qquad \forall\, y \in \mathbb{R}, t \in \mathbb{R}^+ \tag{4.3}$$

where $k_1, k_2 \in \mathbb{R}$ and $k_1 < k_2$. The above condition is a sector constraint for the nonlinearity; we say that $\phi(\cdot,\cdot)$ belongs to the sector $[k_1, k_2]$. This property has the graphical interpretation as shown in Figure 4.2: for all $t \in \mathbb{R}^+$ and $y(t) \in \mathbb{R}$ the graph of $\phi(t, y(t))$ is bounded by the sector defined by two straight lines with the slopes k_1 and k_2.

Summarising the above, the Lur'e system is completely defined by the linear transfer function (4.2) and a given sector $[k_1, k_2]$. Subject to this definition, the problem is to find conditions on $G(s)$ which ensure asymptotic stability for any nonlinearity satisfying the sector condition (4.3). More formally we define *absolute stability*:

Definition 4.1 (Absolute stability) *Given the linear SISO system* (4.1a)–(4.1b) *with* (A, b) *controllable and* (c^{T}, A) *observable and two numbers* $k_1, k_2 \in \mathbb{R}$*,* $k_1 < k_2$*. The equilibrium* $x = 0$ *is absolutely stable if it is globally uniformly asymptotically stable for every function* $\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *belonging to the sector* $[k_1, k_2]$*.*

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[1]Note, that for any given proper rational transfer function $G(s)$ we can find a realisation with the mentioned properties [Che71].

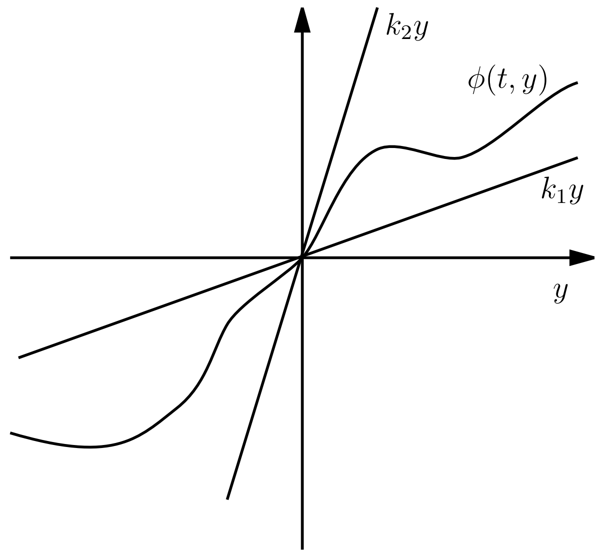


Figure 4.2: Illustration of a sector-bound nonlinearity.

Note that the absolute stability problem is concerned with the stability of an entire family of systems, since $\phi(\cdot, \cdot)$ can be any function satisfying the sector condition (4.3).

<u>Remark:</u> *If the nonlinearity $\phi(\cdot, \cdot)$ is linear in $y(t)$, i.e. $\phi(t, y) = \phi'(t)y(t)$, we can apply a loop transformation to obtain the transformed Lur'e system with*

$$\begin{aligned} \tilde{G}(s) &= \frac{G(s)}{1 + k_1 G(s)} \\ \tilde{\phi}(t, y) &= \phi'(t) - k_1 y(t) \end{aligned}$$

where $\tilde{\phi}(\cdot)$ belongs to the sector $[0, k]$ where $k = \frac{k_2 - k_1}{1 + 2dk_1 + d^2 k_1 k_2}$. For the remainder of this chapter we shall consider Lur'e systems with sector nonlinearities belonging to $[0, k]$.

<u>Remark:</u> *Let $1 + k_0 d = 0$ for some $k_0 \in [0, k]$. For this feedback gain we obtain an unstable closed-loop system since*

$$\lim_{s \to \infty} G_{cl}(s) = \frac{d}{1 + k_0 d}.$$

Using the loop-transformation and Nyquist arguments we can assume without loss of generality that $1 + dk > 0$.

4.2.2 Equivalent stability problems

In this section we show that the absolute stability problem for Lur'e systems is equivalent to the asymptotic stability of a class of switched linear systems.

Let the transfer function $G(s)$ of the linear part of the Lur'e system be given by

$$G(s) \quad = \quad \frac{p_n s^n + \ldots + p_2 s^2 + p_1 s + p_0}{s^n + q_{n-1}s^{n-1} + \ldots + q_2 s^2 + q_1 s + q_0}$$

As realisation $(A, b, c^{\mathsf{T}}, d)$ of the transfer function $G(s)$ we choose the control canonical form:

$$A \quad = \quad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -q_0 & -q_1 & \cdots & -q_{n-2} & -q_{n-1} \end{pmatrix} \tag{4.4}$$

where q_j, $j = 0, \ldots, n-1$, are the coefficients of the characteristic polynomial of A. We refer to that matrix structure as companion form of A. Further,

$$\begin{aligned} b &= \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}^{\mathsf{T}} \\ c^{\mathsf{T}} &= \begin{pmatrix} (p_0 - p_n q_0) & (p_1 - p_n q_1) & \ldots & (p_{n-1} - p_n q_{n-1}) \end{pmatrix} \\ d &= p_n \end{aligned}$$

Consider now the special case where the nonlinearity $\phi(\cdot, \cdot)$ only takes on the extreme values on the sector bound such that

$$\phi(t, y(t)) \quad = \quad \phi'(t) y(t), \qquad \phi'(t) \in \{0, k\} \tag{4.5}$$

Thus, $\phi' : \mathbb{R}_0^+ \to \{0, k\}$ is a piecewise constant function.

Substitution into the equations (4.1) yields

$$\begin{aligned} z(t) &= \frac{\phi'(t)}{1 + \phi'(t) d}\, c^{\mathsf{T}} x(t) \\ \dot{x}(t) &= \left(A - \frac{\phi'(t)}{1 + \phi'(t) d}\, b c^{\mathsf{T}} \right) x(t) \end{aligned} \tag{4.6}$$

The above equations (4.6) and (4.5) describe an autonomous switched linear system with constituent system matrices

$$\begin{aligned} A_1 &= A \\ A_2 &= A - \frac{k}{1+kd}\, bc^{\mathsf{T}} \end{aligned}$$

where the switching signal $\sigma(t) = 1$ for $\phi'(t) = 0$ and $\sigma(t) = 2$ for $\phi'(t) = k$. Note that (4.6) describes a class of switched systems where the constituent system matrices have a difference rank of one, i.e. $rank\{A_1 - A_2\} = 1$.

We show now that the Lur'e system (4.1) is absolutely stable if and only if the switched linear system (4.6) is asymptotically stable for arbitrary switching signals.

Theorem 4.2 *Let the quadruplet* (A, b, c, d) *be a minimal realisation of the proper transfer function* $G(s)$*. Then the Lur'e system* (4.1) *with nonlinearity* $\phi(\cdot,\cdot)$ *belonging to the sector* $[0, k]$ *is absolutely stable if and only if the switched linear system* (2.2) *with* $\mathcal{A} = \left\{A,\ A - \frac{k}{1+kd} bc^{\mathsf{T}}\right\}$ *is asymptotically stable for arbitrary switching signals.*

Proof. The switched system with $\mathcal{A} = \{A_1, A_2\}$ is asymptotically stable for arbitrary switching signals if and only if Σ_{A_1} and Σ_{A_2} have a common Lyapunov function $V(x)$ (c.f. Section 2.4). That implies that $V(x)$ is also a Lyapunov function for all LTI systems $\Sigma_{\bar{A}}$ with $\bar{A} = \alpha A_1 + (1-\alpha)A_2$ for all $\alpha \in [0, 1]$.

On the other hand, at any given time instant t the Lur'e system (4.1) is equivalent to the LTI system

$$\dot{x}(t) = \left(A - \frac{\phi(t)}{1+\phi(t)d} bc^{\mathsf{T}}\right) x(t) \qquad \phi(t) \in [0, k]\,.$$

We need to show that the sets of matrices given by $\left\{\alpha A_1 + (1-\alpha)A_2,\ \alpha \in [0,1]\right\}$ and $\left\{A - \frac{\phi(t)}{1+\phi(t)d} bc^{\mathsf{T}},\ \phi(t) \in [0,k]\right\}$ are equivalent. We require that for any given $\phi(t) \in [0, k]$ there exists an $\alpha \in [0, 1]$ such that

$$\alpha A + (1-\alpha)\left(A - \frac{k}{1+kd} bc^{\mathsf{T}}\right) = A - \frac{\phi(t)}{1+\phi(t)d} bc^{\mathsf{T}}\,.$$

Thus we need to show

$$\begin{aligned} (\alpha - 1)\frac{k}{1+kd} &= -\frac{\phi(t)}{1+\phi(t)d} \\ \alpha &= 1 - \frac{\phi(t)}{1+\phi(t)d}\,\frac{1+kd}{k} \end{aligned}$$

Since $1+\phi(t)d > 0$ for all $\phi(t) \in [0, k]$ there is a unique $\alpha \in [0, 1]$ for every $\phi(t) \in [0, k]$.

This proves that the sets of matrices are equivalent. Hence the common Lyapunov function $V(x)$ for A_1 and A_2 is also a Lyapunov function for the Lur'e system. □

4.3 Spectral condition for a class of strictly positive real transfer functions

In the previous section is shown that the absolute stability of the Lur'e system (4.1) and the asymptotic stability of the switched system (4.6) are equivalent problems. Therefore, stability conditions from either research area are mutually applicable. Naturally, the question arises how stability results obtained for the respective system classes relate. In this chapter we consider the CQLF existence problem which plays a major role in the stability theory for both system classes.

Inspired by the matrix-pencil conditions in Theorem 2.19 for second-order systems and their matrix product formulation of Lemma 2.20 Shorten & Narendra utilised the Circle Criterion to derive a matrix product condition for the existence of a CQLF, [SN03b]. In this chapter we show that a generalised form of that result is applicable to a wide range of conditions obtained for Lur'e systems.

Before we state the main result of this section we shall note two preliminary results that will prove useful later. The first lemma is a well known result from linear algebra:

Lemma 4.3 *[Kai80] Let $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times n}$. Then,*

$$\det[I_n - AB] = \det[I_p - BA] \tag{4.7}$$

where I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$ and $I_p \in \mathbb{R}^{p\times p}$.

The next lemma is a generalisation of a result used in [Kal63]. We shall therefore retain the original notation.

Lemma 4.4 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and $c, b \in \mathbb{R}^{n\times 1}$ and (A, b, c^{T}) a minimal realisation of the strictly proper transfer function $G(s)$. Then, the numerator and denominator polynomials of the rational function,*

$$1 + \mathrm{Re}\left\{G(j\omega)\right\} = \frac{\Gamma(-\omega^2)}{|M(j\omega)|^2} \tag{4.8}$$

are given by

$$\begin{aligned} \Gamma(-\omega^2) &= \left(1 - c^{\mathsf{T}}A\left(\omega^2 I_n + A^2\right)^{-1} b\right)\det\left[\omega^2 I_n + A^2\right], \\ |M(j\omega)|^2 &= \det\left[\omega^2 I_n + A^2\right]. \end{aligned} \tag{4.9}$$

Proof. The transfer function $G(s)$ can be written as (see A.13 in [Kai80])

$$G(s) \quad = \quad c^{\mathsf{T}}\left(sI_n - A\right)^{-1} b \quad = \quad \frac{\det\left[sI_n - A + bc^{\mathsf{T}}\right] - \det\left[sI_n - A\right]}{\det\left[sI_n - A\right]}$$

Substituting this into (4.8) we obtain for the left side

$$\begin{aligned}
1 + \mathrm{Re}\left\{G(j\omega)\right\} \quad &= \quad 1 + \mathrm{Re}\left\{\frac{\det\left[j\omega I_n - A + bc^{\mathsf{T}}\right] - \det\left[j\omega I_n - A\right]}{\det\left[j\omega I_n - A\right]}\right\} \\
&= \quad \mathrm{Re}\left\{1 + \frac{\Big(\det\left[j\omega I_n - A + bc^{\mathsf{T}}\right] - \det[j\omega I_n - A]\Big)\det[-j\omega I_n - A]}{\det[j\omega I_n - A]\ \det[-j\omega I_n - A]}\right\} \\
&= \quad \frac{\mathrm{Re}\Big\{\det\left[\omega^2 I_n + A^2 - bc^{\mathsf{T}}A - j\omega bc^{\mathsf{T}}\right]\Big\}}{\det\left[\omega^2 I_n + A^2\right]} \qquad (4.10)
\end{aligned}$$

Applying an appropriate similarity transformation, we may assume without loss of generality that the rank-one matrix bc^{T} is in one of the Jordan canonical forms:

$$bc^{\mathsf{T}} = \begin{pmatrix} \mu & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad bc^{\mathsf{T}} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Direct computation yields then for the numerator in (4.10)

$$\mathrm{Re}\Big\{\det\left[\omega^2 I_n + A^2 - bc^{\mathsf{T}}A - j\omega bc^{\mathsf{T}}\right]\Big\} = \det\left[\omega^2 I_n + A^2 - bc^{\mathsf{T}}A\right]$$

Hence,

$$\begin{aligned}
1 + \mathrm{Re}\left\{G(j\omega)\right\} \quad &= \quad \frac{\det\left[\omega^2 I_n + A^2 - bc^{\mathsf{T}}A\right]}{\det\left[\omega^2 I_n + A^2\right]} \\
&= \quad \frac{\det\left[\omega^2 I_n + A^2\right]\ \det\left[I_n - \left(\omega^2 I_n + A^2\right)^{-1} bc^{\mathsf{T}}A\right]}{\det\left[\omega^2 I_n + A^2\right]}
\end{aligned}$$

Applying Lemma 4.3 to the numerator yields

$$\begin{aligned}
1 + \mathrm{Re}\left\{G(j\omega)\right\} \quad &= \quad \frac{\det\left[\omega^2 I_n + A^2\right]\left(1 - c^{\mathsf{T}}A\left(\omega^2 I_n + A^2\right)^{-1} b\right)}{\det\left[\omega^2 I_n + A^2\right]} \\
&= \quad \frac{\Gamma(-\omega^2)}{|M(j\omega)|^2}
\end{aligned}$$

$\square$

We can now state the main result of this section. The principal ideas of this lemma appeared first in [SN03b] with variations and extensions in [SCW03].

Lemma 4.5 *Let $G(s)$ be a proper rational transfer function with poles in the open left half-plane and $\{A, b, c, d\}$ be a minimal realisation of $G(s)$ such that $G(s) = c^{\mathsf{T}}(sI - A)^{-1}b + d$, and let $K \in \mathbb{R}$ satisfy $K + d > 0$. Then the following are equivalent:*

(i) $K + \mathrm{Re}\{G(j\omega)\} > 0, \qquad \forall\, \omega \in \mathbb{R}$,

(ii) the matrices $A_1 = A$ and $A_2 = \left(A - \frac{bc^{\mathsf{T}}}{K+d}\right)$ are Hurwitz and the matrix product A_1A_2 has no negative real eigenvalue.

Proof. The first statement in the lemma can be written as

$$\begin{aligned} K + \mathrm{Re}\{G(j\omega)\} &> 0\,, \qquad \forall\, \omega \in \mathbb{R} \\ K + d + \mathrm{Re}\left\{c^{\mathsf{T}}(j\omega I_n - A)^{-1}b\right\} &> 0 \end{aligned} \tag{4.11}$$

Applying Lemma 4.4 we obtain

$$\begin{aligned} K + d - 1 + \frac{\left(1 - c^{\mathsf{T}}A\left(\omega^2 I_n + A^2\right)^{-1}b\right)\det\left[\omega^2 I_n + A^2\right]}{\det\left[\omega^2 I_n + A^2\right]} &> 0 \\ K + d - c^{\mathsf{T}}A(\omega^2 I_n + A^2)^{-1}b &> 0 \end{aligned}$$

which is equivalent to

$$\det\left[K + d - c^{\mathsf{T}}A(\omega^2 I_n + A^2)^{-1}b\right] > 0.$$

Applying Lemma 4.3 yields

$$\begin{aligned} \det\left[(K+d)I_n - (\omega^2 I_n + A^2)^{-1}bc^{\mathsf{T}}A\right] &> 0 \\ \det\left[(\omega^2 I_n + A^2)^{-1}\right]\,\det\left[(K+d)(\omega^2 I_n + A^2) - bc^{\mathsf{T}}A\right] &> 0 \\ \det\left[(\omega^2 I_n + A^2)^{-1}\right](K+d)^n\,\det\left[\omega^2 I_n + A^2 - \frac{1}{K+d}bc^{\mathsf{T}}A\right] &> 0 \end{aligned}$$

Since A is Hurwitz, all real eigenvalues of A^2 are positive. Thus $\det\left[\omega^2 I_n + A^2\right] > 0$ for all $\omega \in \mathbb{R}$. With $K + d > 0$ we obtain

$$\det\left[\omega^2 I_n + \left(A - \frac{1}{K+d}bc^{\mathsf{T}}\right)A\right] > 0. \tag{4.12}$$

Let $A_1 = A$ and $A_2 = A - \frac{1}{K+d}bc^{\mathsf{T}}$ and suppose A_1A_2 has no negative real eigenvalue. Then inequality (4.11) holds for all $\omega \in \mathbb{R}$, thus $K + \mathrm{Re}\{G(j\omega)\} > 0\ \forall\, \omega \in \mathbb{R}$.

On the other hand, if the product A_2A_1 has a negative real eigenvalue then there exists some $\omega_0^2 > 0$ for which $\det\left[\omega_0^2 I_n + A_2A_1\right] = 0$ and condition (4.12) does not hold, i.e. condition (4.11) is violated if A_1A_2 has the eigenvalue $-\omega_0^2$. $\square$

4.4 Implications for classical stability results

A number of classical stability conditions for Lur'e systems are stated as strictly positive real (SPR) conditions of some transfer function $H(s)$.

Definition 4.6 (Positive real transfer function) *[NT73] A function $H(s)$ is positive real if*

(i) $H(s)$ is real for $s \in \mathbb{R}$,

(ii) $\mathrm{Re}\{H(s)\} \geq 0$ for all $\mathrm{Re}\{s\} > 0$.

$H(s)$ is strictly positive real if and only if $H(s-\varepsilon)$, $\varepsilon > 0$ is positive real [Tay74].

Determining whether a transfer function is strictly positive real is not always straight forward (see [NT73] or [IT87] for details). However the following result shall be sufficient for the cases of our discussion.

Theorem 4.7 *[IT87] Let $H(s)$ be a rational transfer function with poles in the open left half-plane and relative degree $n^* = 0$, i.e. numerator and denominator have the same degree. Then $H(s)$ is strictly positive real if and only if*

$$\mathrm{Re}\big\{H(j\omega)\big\} \quad > \quad 0 \qquad \text{for all } \omega \in \mathbb{R}\,. \tag{4.13}$$

Applying Lemma 4.5, we can formulate a new condition for strict positive real transfer functions.

Theorem 4.8 *Let (A, b, c, d) be a minimal realisation of the proper transfer function $H(s)$ with poles in the open left half-plane and $d > 0$. Then $H(s)$ is strictly positive real if and only if $\left(A - \frac{1}{d}bc^{\mathsf{T}}\right)$ is Hurwitz and the matrix product $A\left(A - \frac{1}{d}bc^{\mathsf{T}}\right)$ has no negative real eigenvalue.*

This theorem follows immediately from Theorem 4.7 and Lemma 4.5. A generalisation of the result for the case $d = 0$ can be found in [SK04].

Many conditions derived for Lur'e systems are based on the positive realness of some transfer function. We shall now employ the Lemma 4.5 to obtain eigenvalue conditions for results formulated in terms of strictly positive real functions.

One of the most fundamental results derived for Lur'e systems is the Kalman Yakubovic Popov (KYP) lemma. Note that a number of different versions of the KYP lemma have been derived by various authors (see [NA89, Vid93] for an overview).

Theorem 4.9 (KYP lemma) *Given a Hurwitz matrix $A \in \mathbb{R}^{n\times n}$ and vectors $b, c \in \mathbb{R}^n$ such that the pair (A, b) is controllable and given scalars $k \geq 0$ and $\varepsilon > 0$, and a positive definite matrix $Q \in \mathbb{R}^{n\times n}$. Then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ satisfying*

$$A^{\mathsf{T}}P + PA = -qq^{\mathsf{T}} - \varepsilon Q \tag{4.14a}$$

$$Pb - c^{\mathsf{T}} = \sqrt{\gamma}q \tag{4.14b}$$

if and only if

$$k + \mathrm{Re}\left\{c^{\mathsf{T}}(j\omega I_n - A)^{-1}b\right\} > 0 \qquad \forall\, \omega \in \mathbb{R}. \tag{4.15}$$

The system considered in the KYP lemma satisfies the assumptions of Lemma 4.5. Therefore we can reformulate the KYP lemma as eigenvalue condition.

Theorem 4.10 (Spectral version of the KYP lemma) *Given a Hurwitz matrix $A \in \mathbb{R}^{n\times n}$ and vectors $b, c \in \mathbb{R}^n$ such that the pair (A, b) is controllable and given scalars $k \geq 0$ and $\varepsilon > 0$, and a positive definite matrix $Q \in \mathbb{R}^{n\times n}$. Then there exists a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$ satisfying* (4.14) *if and only if the matrix product $A\left(A - \frac{1}{k}bc^{\mathsf{T}}\right)$ has no negative real eigenvalue.*

The KYP lemma plays a key role in the proof of several stability results for Lur'e systems. One of the best known stability condition is the Circle Criterion.

Theorem 4.11 (Circle Criterion) *Consider the Lur'e system* (4.1) *with the transfer function $G(s)$ and sector nonlinearity $\phi(t, y)$ belonging to the sector $[0, k]$. Then system* (4.1) *is absolutely stable if $G(s)$ is a stable transfer function and*

$$\frac{1}{k} + \mathrm{Re}\big\{G(j\omega)\big\} > 0 \qquad for\ all\ \omega \in \mathbb{R}. \tag{4.16}$$

It follows from Lemma 4.5 that the condition (4.16) holds if and only if the matrices $A_1 = A$ and $A_2 = \left(A - \frac{k}{1+kd}bc^{\mathsf{T}}\right)$ are Hurwitz and the matrix product A_1A_2 has no negative real eigenvalue [SN03b]. Since the matrices A_1, A_2 represent the closed-loop system we can immediately incorporate loop-transformations for arbitrary sector bounds by choosing $A_1 = \tilde{A} - \frac{k_1}{1+k_1 d} bc^\mathsf{T}$ and $A_2 = \tilde{A} - \frac{k_2}{1+k_2 d} bc^\mathsf{T}$, where $\tilde{A}$ is the system matrix of the transformed transfer function $\tilde{G}(s) = \frac{G(s)}{1-k_1 G(s)}$. The Criterion takes then the following form.

Theorem 4.12 (Spectral version of the Circle Criterion) *Consider the Lur'e system* (4.1) *with the transfer function $G(s)$ and sector nonlinearity $\phi(t, y)$ belonging to the sector $[k_1, k_2]$ with $k_1, k_2 \in \mathbb{R}$. Then system* (4.1) *is absolutely stable if A_1 and A_2 are Hurwitz and the matrix product $A_1 A_2$ has no negative real eigenvalues.*

Sufficiency of the Circle Criterion for the existence of a CQLF was directly shown in [NG64] by means of the KYP lemma. Necessity was first established in [Wil73] using indirect arguments. With Lemma 4.4 we can derive an alternative proof for necessity:

Suppose condition (4.16) is violated, i.e.

$$\frac{1}{k} + \mathrm{Re}\{G(j\omega)\} \quad = \quad 0$$

for some ω_0. Then by Lemma 4.4

$$\begin{aligned} \det\left(\omega_0^2 I + A_2 A_1\right) &= 0 \\ \det\left(\omega_0^2 A_2^{-1} + A_1\right) &= 0 \end{aligned} \tag{4.17}$$

The latter follows since A_2 is Hurwitz. This implies that the matrix pencil $\sigma_\gamma[A_1^{-1}, A_2] = A_1^{-1} + \gamma A_2$ is singular for $\gamma = \omega_0^2$. However, it is a necessary condition for the existence of a CQLF that the matrix pencil $\sigma_\gamma[A_1^{-1}, A_2]$ is Hurwitz for all $\gamma > 0$.

Multiplier criteria

For various classes of the Lur'e system (4.1) stability conditions have been derived in terms of the frequency response $G(j\omega)$ involving the use of multipliers, e.g. [Pop61, ZF68, CN68]. These conditions typically require the existence of some transfer function $M(s)$, called *multiplier*, such that

$$\mathrm{Re}\big\{M(j\omega)G(j\omega)\big\} \quad > \quad 0 \qquad \forall\, \omega \in \mathbb{R}.$$

to establish stability of different classes of the Lur'e system.

In many cases $M(s)G(s)$ is a bi-proper real rational transfer function with poles in the open left half-plane where $\left(\bar{A}, \bar{b}, \bar{c}, \bar{d}\right)$ is a minimal realisation of $M(s)G(s)$ with $\bar{d} > 0$. For these cases we can apply Lemma 4.5 to obtain eigenvalue conditions for stability. The best known multiplier criterion is the Popov criterion:

Theorem 4.13 (Popov Criterion) *Let $G(s) = c^{\mathsf{T}}\left(sI - A\right)^{-1} b$ be the transfer function of the system* (4.1) *and consider the time-invariant feedback nonlinearity $\phi(\cdot)$ belonging to the open sector $(0, k)$. Then the feedback system is absolutely stable if there exists a number $\alpha > 0$ such that $H(s) = \frac{1}{k} + (1 + j\alpha\omega)G(s)$ is strictly positive real.*

Applying Theorem 4.8 on strict positive realness and Lemma 4.5 we can reformulate the Popov criterion.

Theorem 4.14 (Spectral version of the Popov Criterion) *Consider the system* (4.1) *where $d = 0$ and with the time-invariant nonlinearity $\phi(\cdot)$ belonging to the open sector $(0, k)$. Let $H(s) = \frac{1}{k} + (1 + j\alpha\omega)G(s)$ and let $\left(\bar{A}, \bar{b}, \bar{c}, \bar{d}\right)$ be a minimal realisation of $H(s)$. Then system* (4.1) *is absolutely stable if there is a number $\alpha > 0$ such that $\bar{d} \neq 0$, the poles of $H(s)$ are in the open left half-plane, the matrices $A_1 = \bar{A}$ and $A_2 = \bar{A} - \frac{1}{\bar{d}}\bar{b}\bar{c}^{\mathsf{T}}$ are Hurwitz, and the matrix product $A_1 A_2$ has no negative real eigenvalues.*

Note, that in all the above classic frequency-domain conditions, the test of an infinite number of points $\omega \in \mathbb{R}$ has been replaced by a single eigenvalue calculation in the time-domain.

Relation to Optimal control

An alternative condition for the existence of the quadratic Lyapunov function of the KYP lemma can be found in the context of optimal control. The condition is given in form of eigenvalue constraints on the Hamiltonian matrix associated with the solution of the algebraic Ricatti equation [BEFB94]. There exists a Lyapunov function of the form (4.14) for the Lur'e system (4.1) if and only if the Hamiltonian matrix

$$H = \begin{pmatrix} A - \frac{k}{2}bc^{\mathsf{T}} & \frac{1}{2}bb^{\mathsf{T}} \\ -\frac{k^2}{2}cc^{\mathsf{T}} & -A^{\mathsf{T}} + \frac{k}{2}cb^{\mathsf{T}} \end{pmatrix}$$

has no (non-zero) purely imaginary eigenvalues.

This condition has a similar form as the spectral condition for KYP lemma in Theorem 4.10. This suggests that the eigenvalues of the Hamiltonian matrix H and the eigenvalues of the matrix product are related. Indeed the following equivalence can be established [CS04]

$$\det\left(\lambda I_{2n} - H\right) \quad = \quad \det\Big(\lambda^2 I_n - A\Big(A - \frac{1}{k}bc^\mathsf{T}\Big)\Big) .$$

From that follows that the Hamiltonian matrix H has a purely imaginary eigenvalue $j\omega \neq 0$ if and only if the matrix product $A\left(A - \frac{1}{k}bc^\mathsf{T}\right)$ has a real negative eigenvalue $-\omega^2$.

4.5 Implications for switched systems

In the previous section the implications of Lemma 4.5 for frequency inequality conditions for Lur'e type systems have been outlined. On the other hand, the principle ideas of Lemma 4.5 have been initially used to derived necessary and sufficient conditions for the existence of a CQLF for a class of switched systems [SN03a]. Moreover, the eigenvalues of the matrix product A_1A_2 and the eigenvalue locus of matrix-pencil $\sigma_\alpha[A_1, A_2]$ are closely related. In this section a number of results are collected to summarise the relation of the matrix products, matrix-pencils and the existence of a CQLF for the LTI system Σ_{A_1} and Σ_{A_2}.

In [Loe76] and [BBP78] the following is shown.

Theorem 4.15 *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix. Then any quadratic Lyapunov function $V(x)$ for the LTI system Σ_A is also a Lyapunov function for $\Sigma_{A^{-1}}$.*

Recall from Lemma 2.12 that if there exists a common Lyapunov function for the systems Σ_{A_1} and Σ_{A_2} then the matrix pencil $\sigma_\gamma[A_1, A_2]$ is Hurwitz for all $\gamma > 0$. From Theorem 4.15 it follows that it is also necessary for the existence of a CQLF for Σ_{A_1} and Σ_{A_2} that the matrix pencil $\sigma_\gamma[A_1^{-1}, A_2]$ is Hurwitz for all $\gamma > 0$.

Lemma 4.16 *[SN98a] Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$. A necessary condition for the existence of a CQLF for the systems Σ_{A_1} and Σ_{A_2} is that the matrix pencils $\sigma_\gamma[A_1, A_2]$ and $\sigma_\gamma[A_1^{-1}, A_2]$, and equivalently $\sigma_\gamma[A_2, A_1]$ and $\sigma_\gamma[A_2^{-1}, A_1]$, are Hurwitz.*

Note that the conditions of the above lemma are also sufficient for the existence of a CQLF when $A_1, A_2 \in \mathbb{R}^{2\times 2}$ (Theorem 2.19). For this system class Lemma 2.20 establishes that the matrix pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz if and only if the matrix product $A_1^{-1}A_2$ has no negative real eigenvalue. Thus there exists a CQLF for Σ_{A_1} and Σ_{A_2} with $A_1, A_2 \in \mathbb{R}^{2\times 2}$ if and only if A_1, A_2 are Hurwitz and the matrix products $A_1^{-1}A_2$ and A_1A_2 have no negative real eigenvalue.

This result can be generalised for a class of systems with $A_1, A_2 \in \mathbb{R}^{n\times n}$ using a variation of the proof of Lemma 4.5 and its relation to the Circle Criterion [SMCC04].

Theorem 4.17 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ and* $\mathrm{rank}\{A_2 - A_1\} = 1$. *Then the LTI systems Σ_{A_1} and Σ_{A_2} have a CQLF if and only if the matrix product A_1A_2 has no negative real eigenvalue.*

Note that for matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ with a difference of rank 1, only one of the two matrix products needs to be checked. In fact, it has been shown for systems with *rank*$\{A_1 - A_2\} = 1$ that system class the product $A_1^{-1}A_2$ cannot have any negative real eigenvalues and that the matrix-pencil $\sigma_\alpha[A_1, A_2]$ is always Hurwitz [LSC02].

Assuming that A_1, A_2 are Hurwitz, it follows from (4.17) that the matrix product A_1A_2 has no negative real eigenvalue if and only if $A_1+\gamma A_2^{-1}$ is non-singular. Further, by Theorem 4.16 it is necessary for the existence of a CQLF that the matrix-pencil $\sigma_\gamma[A_1, A_2^{-1}]$ is Hurwitz. Hence, a necessary and sufficient condition for the existence of a CQLF for the switched system (4.6) is that $A_1 + \gamma A_2^{-1}$ is Hurwitz for all $\gamma \geq 0$. This has a somewhat surprising analogy to Aizerman's famous conjecture [Aiz49]. The corresponding formulation of this conjecture for our system class is that the system (4.6) is absolutely stable if $A_1 + \gamma A_2$ is Hurwitz for all $\gamma \geq 0$. It is well known that the Aizerman Conjecture is false.

From the above discussion we can conclude the following summarising corollary.

Corollary 4.18 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ and* $\mathrm{rank}\{A_2 - A_1\} = 1$. *Then the following statements are equivalent:*

(i) there exists a CQLF for the LTI systems Σ_{A_1}, Σ_{A_2}, $\Sigma_{A_1^{-1}}$ and $\Sigma_{A_2^{-1}}$;

(ii) the matrix products A_1A_2, $A_1^{-1}A_2$, A_2A_1 and $A_2^{-1}A_1$ have no negative real eigenvalues;

(iii) the matrix pencils $A_1^{-1}+\gamma A_2$, $A_1+\gamma A_2$, $A_2^{-1}+\gamma A_1$, and $A_2+\gamma A_1$ are Hurwitz for all $\gamma \geq 0$.

4.6 Application of the generalised SPR condition

In this section we consider some examples to demonstrate the application of some of the results in this chapter. It is shown that the application of Lemma 4.5 can simplify some control-design tasks. Moreover, we show that the graphical interpretation of the frequency inequality allows the application of Lemma 4.5 to further design criteria for LTI system.

Not least due to its graphical interpretation the Circle Criterion proves very useful for the design of linear controllers for nonlinear systems. Let $D(k_1, k_2)$ denote the disk in the complex plane with diameter on the real axes defined by the interval $\left[-\frac{1}{k_1}, -\frac{1}{k_2}\right]$. Then the graphical version of the Circle Criterion is given by:

Theorem 4.19 *The Lur'e system* (4.1) *with nonlinearity* $\phi(\cdot,\cdot)$ *belonging to the sector* $[k_1, k_2]$ *and* $G(s) = c^{\mathsf{T}}(sI - A)^{-1}b + d$ *with* ν *poles in the right half-plane is absolutely stable if*

(i) $0 < k_1 < k_2$*: the Nyquist plot of* $G(s)$ *lies entirely outside the disk* $D(k_1, k_2)$ *and encircles it* ν *times counter-clock-wise;*

(ii) $0 = k_1 < k_2 : \nu = 0$ *and the Nyquist plot of* $G(s)$ *lies entirely to the right of the line* $\mathrm{Re}\{s\} = -\frac{1}{k_2}$*;*

(iii) $k_1 < 0 < k_2 : \nu = 0$ *and the Nyquist plot of* $G(s)$ *lies entirely inside the disk* $D(k_1, k_2)$*.*

The main difference of the formulation of the stability results in terms of the frequency response and the matrix product, lies in the fact that the classical stability results are formulated in terms of the open-loop transfer function $G(s)$ whereas the matrix-product conditions involves the closed loop system matrices $A - k_i bc^{\mathsf{T}}$. The original formulation of the stability conditions proves very useful when applying classical frequency-domain design methods like loop-shaping, the eigenvalue condition could have advantages to determine parameter-constraints.

Example 4.1 *Consider the Lur'e system with the linear transfer function given by*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; d = 0.$$

and nonlinearity $\phi(\cdot,\cdot)$ *belonging to the sector* $[k_1, k_2]$*. The objective is to determine pairs of parameters* (k_1, k_2) *such that the Lur'e system is absolutely stable.*

Applying the condition of Theorem 4.12, let $A_1 = A - k_1 bc^{\mathsf{T}}$ *and* $A_2 = A - k_2 bc^{\mathsf{T}}$*. Then we obtain for the matrix product*

$$A_1 A_2 \quad = \quad \begin{pmatrix} -1-k_2 & -2 \\ 2+2k_2 & 3-k_1 \end{pmatrix}.$$

We can now calculate the eigenvalues of that matrix product for values (k_1, k_2) *in the parameter space. Figure 4.3 shows the properties of the eigenvalues of* A_1A_2 *over the parameter space. For parameter pairs that yield a matrix* A_1A_2 *with negative real eigenvalues are dark shaded. The white area indicates parameter combinations for which the Circle Criterion holds. From this diagram we can directly read off, parameter combinations that ensure the stability of the Lur'e system.*

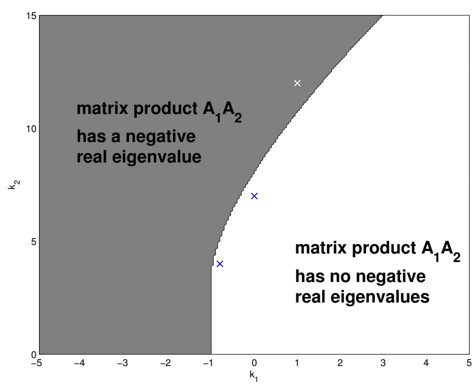


Figure 4.3: Sector bounds for which the matrix product A_1A_2 in Example 4.1 has negative real eigenvalues

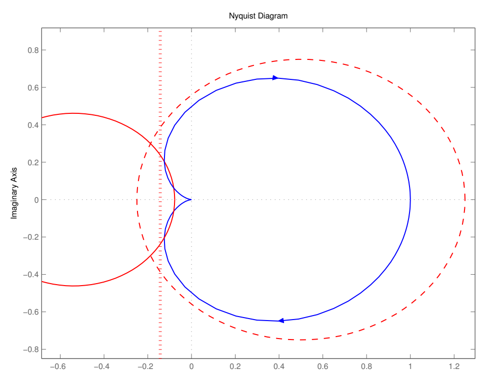


Figure 4.4: Nyquist plot of $G(j\omega)$ for Example 4.1 with circles given for nonlinearities belonging to two different sectors.

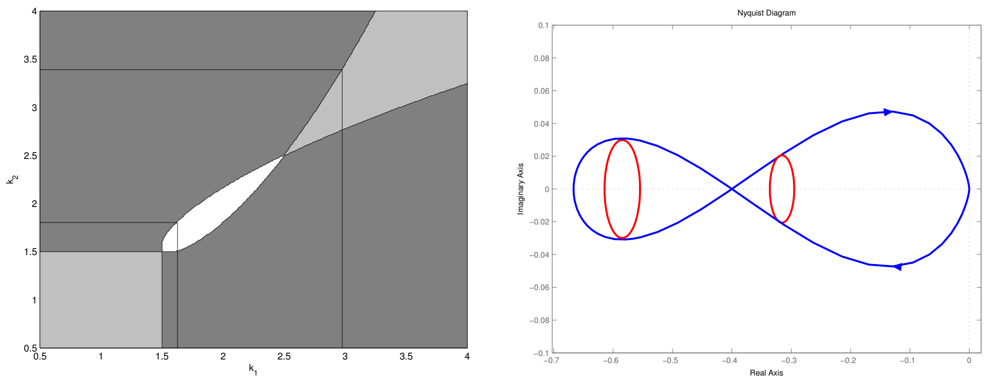
Figure 4.4 shows the Nyquist plot of $G(j\omega)$ *to verify those findings. For comparison with the classical Circle Criterion we choose the three sectors* $[-0.8, 4]$, $[0, 7]$ *and* $[1, 12]$*. As indicated in Figure 4.3, the first two points yield stable systems and for the last sector bounds there exists no CQLF. For the first sector we have to apply case* (iii) *of the Circle Criterion, i.e. the Nyquist plot lies inside the disk* $D(-0.8, 4)$ *(dashed line). In the second case the Nyquist plot lies to the right of the (dotted) line* $\mathrm{Re}\{s\} = -\frac{1}{7}$*. Only the third case violates the Circle Criterion since the Nyquist plot intersects the disk* $D(1, 12)$ *(solid line). These findings for the three cases marked in Figure 4.3 are confirmed by the matrix product condition.* □

Example 4.2 *(taken from [Kha96])*

Consider the Lur'e system (4.1) *with the unstable linear transfer function*

$$G(s) = \frac{4}{(s-1)(s+\frac{1}{2})(s+\frac{1}{3})}$$

Again, we are interested in determining sectors $[k_1, k_2]$ *for which the system is stable. Choosing some minimal realisation* $\{A, b, c\}$ *of* $G(s)$*, Theorem 4.12 provides a straight*



(a) (b)

Figure 4.5: Stability evaluation for Example 4.2. Part (a) shows the results of the evaluation of the matrix product. For parameter combinations in the dark grey area the matrix product has negative real eigenvalues. In the light grey areas one of the matrices A_1, A_2 is non-Hurwitz. Only sector bounds of the white area satisfy the conditions of Theorem 4.12. Part (b) shows the Nyquist plot of $G(s)$. Two parameter combinations are chosen for verify the results with the Circle Criterion. The sector bound $[3, 3.5]$ correspond to the disk in the right lope, the disk of the sector $[1.6, 1.8]$ lies in the left lope.

forward answer. Let $A_1 = A - k_1 bc^{\mathsf{T}}$ *and* $A_2 = A - k_2 bc^{\mathsf{T}}$*. The eigenvalue condition of Theorem 4.12 for different sector bounds is depicted in Figure 4.5a. The dark shaded areas denote parameter combinations* (k_1, k_2) *for which the matrix product* A_1A_2 *has real negative eigenvalues. For the light shaded areas the product has no negative real eigenvalue, but either* A_1 *or* A_2 *is not Hurwitz and therefore the system has no common quadratic Lyapunov function. Only for parameters in the white area the requirements of Theorem 4.8 are satisfies and thus there exists a CQLF.*

Figure 4.5b shows the Nyquist plot of $G(j\omega)$*. From the Circle Criterion we know that the Nyquist plot must encircle the disk* $D(k_1, k_2)$ *once counter clockwise, since the* $G(j\omega)$ *has one pole in the right half-plane. Hence the disk has to be inside the left lope.*

4.6.1 Other Nyquist-based circle conditions

The graphical interpretation of the Circle Criterion in Theorem 4.19 suggests that the eigenvalue condition in Theorem 4.12 is applicable to other conditions expressed by the location of the Nyquist plot and some circle in the complex plane. Such conditions appear for example in the sensitivity analysis of linear systems [ÅPH98].

The following example demonstrates how the eigenvalue condition in Theorem 4.12 can be applied to design a PI-controller for a nonlinear plant, subject to stability and sensitivity constraints. The example is taken from [**?**] where similar analysis is used to design a switched controller for an ABS system of an automobile.

Example 4.3 *Consider the system in Figure 4.6. The plant is given by the Lur'e system with linear part* $G_p(s) = \frac{1}{(s+1)^3}$ *and nonlinearity* $\phi(\cdot,\cdot)$ *belonging to the sector* $[1, 4]$. *The objective is to design a PI-controller such that the closed loop system is stable and has a sensitivity to modelling errors of* $M_s = \frac{1}{0.6}$.

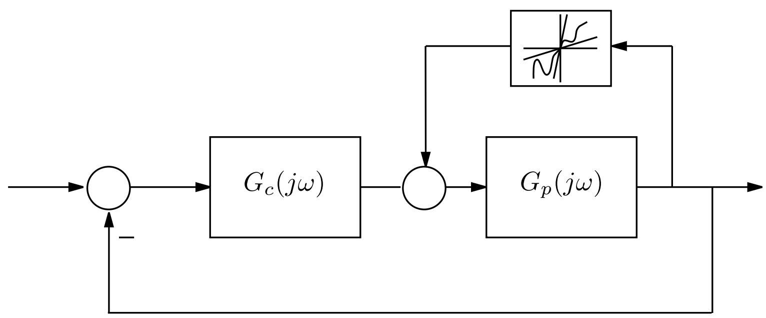


Figure 4.6: Control system of Example 4.3.

The transfer function of the PI-controller is given by

$$G_c(s) = K_p + \frac{K_i}{s}$$

where $K_p, K_i > 0$ are the design parameters.

We shall now apply the matrix product condition parameterised by K_p and K_i to obtain a controller that satisfies both constraints.

Stability analysis The closed loop system in Figure 4.6 is equivalent with the Lur'e system with nonlinear feedback belonging to the sector $[1, 4]$ and linear transfer function

$$\begin{aligned} G(s) &= \frac{G_p(s)}{1+G_p(s)G_c(s)} \\ &= \frac{s}{s^4+3s^3+3s^2+(1+K_p)s+K_i} \end{aligned}$$

Choosing the control canonical realisation $\{A, b, c\}$ of $G(s)$ we can apply Theorem 4.12. We evaluate the matrix product $\left(A - bc^{\mathsf{T}}\right)\left(A - 4bc^{\mathsf{T}}\right)$ for values of K_p and K_i we are interested in. The matrix product has negative real eigenvalues for parameter combinations (K_i, K_p) above the dashed line in Figure 4.7. Thus we obtain controllers that satisfy the stability constraint by the parameter space below the dashed line.

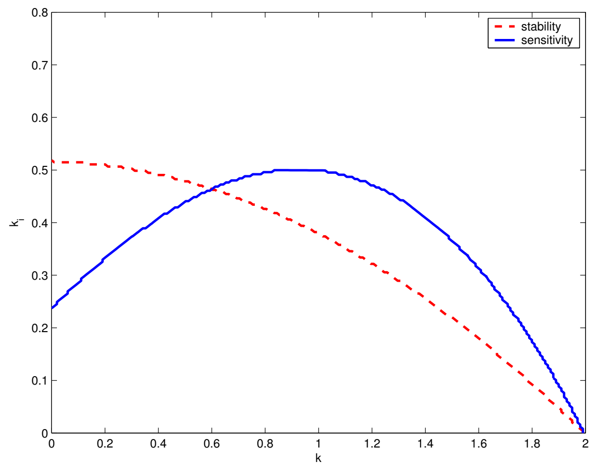


Figure 4.7: Parameter constraints for the controller in Example 4.3. Parameter combinations below the dashed line satisfy the stability constraint, combinations below the solid line satisfy the sensitivity constraint.

Sensitivity analysis The sensitivity constraint requires that the open-loop transfer function $L(s) = G_cG_p = \frac{K_ps+K_i}{s(s+1)^3}$ does not intersect the circle with radius $R_s = 0.6$ around the point $(-1, 0)$, [ÅPH98]. Let $\{A_L, b_L, c_L\}$ be a minimal realisation of $L(s)$. By geometric equivalence this constraint is satisfied when the matrix product $\left(A_L - \alpha b_L c_L^{\mathsf{T}}\right)\left(A_L - \beta b_L c_L^{\mathsf{T}}\right)$ with $\alpha = \frac{1}{1.6}$ and $\beta = \frac{1}{0.4}$ has no negative eigenvalue. Again we evaluate the eigenvalues of the product for the chosen set of controller parameters. The area below the solid line in Figure 4.7 denotes parameter sets for which the sensitivity constraint is satisfied.

We can now read off controller parameters that satisfy both conditions. For performance reasons we would usually aim to maximise the integrator gain [ÅPH98]. Therefore, we choose the controller at the intersection of both constraints: $K_p =$

$0.6, K_i = 0.46$. Figure 4.8 shows the Nyquist plot of the open-loop transfer function $L(j\omega)$ for the chosen controller. The plot does not intersect the circle of radius $R_s = 0.6$ and hence the sensitivity constraint is satisfied. Figure 4.9 confirms that the Nyquist plot $G(j\omega)$ for the chosen controller does not intersect the critical circle and hence a common quadratic Lyapunov function exists.

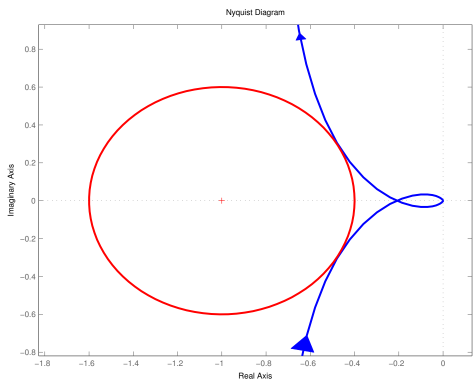


Figure 4.8: Sensitivity plot for the controller in Example 4.3

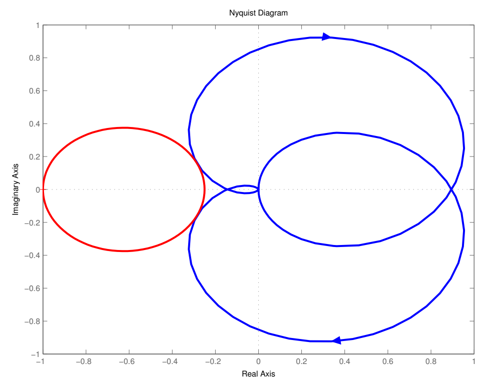


Figure 4.9: Stability for the controller in Example 4.3

4.7 Conclusions

In this chapter we established a connection between a class of switched systems and SISO Lur'e systems. It is shown that the problem of absolute stability is equivalent to the asymptotic stability of the corresponding switched system for arbitrary switching. This equivalence enables stability results obtained for the respective system classes to be applied interchangeably. The main result of this chapter relates the frequency-response inequality $K + \text{Re}\{G(s)\} > 0$ to eigenvalue properties of the corresponding matrix product $A_1 A_2$. This equivalence casts a new light on the large number of results that have been derived in terms of this inequality. The implications and possible interpretations of the reformulation of these conditions leave room for further research.

Chapter 5

Existence of periodic motion and absolute stability

In this chapter we consider the stability of a class of switched linear single-input single-output systems whose asymptotic stability is equivalent to the absolute stability of the corresponding Lur'e system. The approach is adopted from a stability conjecture first posed in [PT74] which aims to predict the existence of periodic motion. The validity of the conjecture is examined and several questions arising from it are analysed. Based on this analysis a necessary and sufficient condition for the existence of periodic motion is derived. It is shown that a numerical approximation of this result can yield good estimates for the existence of periodic motion.

5.1 Introductory remarks

In the previous chapter the equivalence of the absolute stability of SISO Lur'e systems and the asymptotic stability of the corresponding switched linear system has been established (Theorem 4.2). It has been shown that this equivalence can be exploited to gain new insights into the analysis of either system class. In this chapter we shall use the frequency-domain description of the Lur'e system to obtain non-quadratic stability conditions for the two system classes. In particular we shall consider the existence of periodic motion and its relation to the stability of the considered system. Clearly, if there exists a periodic solution $x(t, t_0, x_0, \sigma) = x(t+T, t_0, x_0, \sigma)$ for the autonomous switched linear system (2.4) the system is not asymptotically stable. However, our analysis reveals that the non-existence of such periodic motion is of some significance for the absolute stability of the Lur'e system. This insight is exploited to derive conditions that approximate the largest symmetric sector $[-k, k]$ for which the Lur'e system is absolutely stable.

In this chapter we continue to analyse the behaviour of switched systems with two constituent systems

$$\dot{x} \quad = \quad A(t)x \qquad A(t) \in \mathcal{A} = \{A_1, A_2\} \subset \mathbb{R}^{n \times n} \tag{5.1}$$

where A_1 and A_2 are in companion form (4.4) (or simultaneously similar to companion form). We shall consider stability for arbitrary switching signals $\sigma(\cdot)$ as in Definition 2.1.

For the analysis in this chapter we choose a frequency-domain approach to exploit characteristics of the frequency-spectrum of the periodic signals. As in Section 4.2 we shall therefore describe the switched system (5.1) by the corresponding Lur'e system in Figure 5.1. Let $A = \frac{1}{2}\big(A_2 + A_1\big)$ and choose vectors b, c and a number $k > 0$ that satisfy $kbc^{\mathsf{T}} = \frac{1}{2}\big(A_2 - A_1\big)$. Then by Theorem 4.2 the asymptotic stability of the switched system (5.1) is equivalent to the absolute stability of the Lur'e system with

$$G(s) \quad = \quad c^{\mathsf{T}}(sI - A)^{-1}b \tag{5.2a}$$

$$z(t) \quad = \quad \phi(t)y(t) \tag{5.2b}$$

where the time-varying feedback gain $\phi(\cdot)$ belongs to the sector $[-k, k]$.

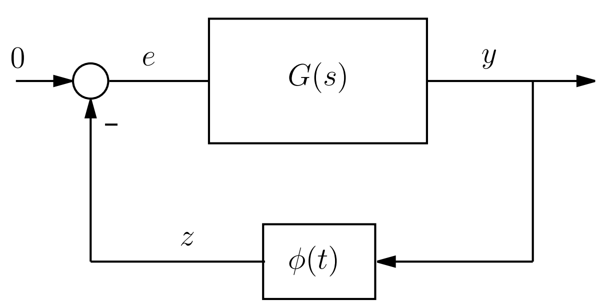


Figure 5.1: Feedback representation of the switched system (5.2).

Some remarks on the considered system classes are in order:

(i) Note that system (5.2) is equivalent to the switched system (5.1) if $\phi(t) =$

$k(2\sigma(t) - 3)$. Since the value-set of $\sigma(\cdot)$ is simply the index set $\mathcal{I}$ whose values can be chosen arbitrarily, the set of switching signals $\sigma(\cdot)$ and the set of feedback modulation signals $\phi(\cdot)$ are essentially the same. To keep notation simple we shall refer to $\mathcal{S}$ as the set of admissible switching signals $\sigma(\cdot)$ and the set of feedback modulations $\phi(\cdot)$ in (5.2b) interchangeably.

(ii) The Lur'e system (5.2) is only defined for the time-varying feedback gain $\phi(\cdot)$. Commonly the Lur'e system is defined for more general nonlinearities for which $z(t) = \phi\big(t, y(t)\big)$ and $\phi(\cdot, \cdot)$ belongs to the sector $[-k, k]$. However, as established in Section 4.2, for the analysis of absolute stability it is necessary and sufficient to consider the class of time-varying feedback gains.

(iii) The stability results in this chapter hold simultaneously for switched systems of the form (5.1) and for Lur'e systems (5.2). Note, that we consider Lur'e systems with strictly proper transfer functions (5.2a) such that $\lim_{\omega\to\infty} G(j\omega) = 0$.

We pursue two main objectives in this chapter. Firstly we shall investigate how to determine the stability boundary of system (5.2) in terms of the parameter k, i.e. given the matrix A and vectors b, c we want to find the largest sector $[-k, k]$ for which the Lur'e system (5.2) is absolutely stable. The second line of inquiry aims to identify a subclass of switching signals from which we can derive necessary and sufficient conditions for the asymptotic stability of the switched system (5.1) for arbitrary switching signals.

The stability analysis in this chapter is motivated by a conjecture proposed independently in [PT74] and later in [Sho96]. Instead of finding a common Lyapunov function, the authors adopt an approach similar to the describing function approach [Moh91] in order to determine the possibility of periodic motion. Although, the examination in this chapter shows that this conjecture is in general false, a number of interesting questions arise from its analysis. Based on the discussion of these issues we develop a sufficient condition for the absolute stability of the Lur'e system.

The chapter is organised as follows. In the next section we introduce the approach and the stability conjecture in [PT74]. In Section 5.3 the validity of the conjecture is examined and several questions arising from the conjecture are discussed. Based on these findings a sufficient condition for absolute stability is derived in Section 5.4. This condition is based on the evaluation of the eigenvalues of infinite-dimensional matrices. In Section 5.5 we analyse the effects of approximating the infinite-dimensional matrices by finite truncations. In Section 5.6 some observations on the behaviour of switched linear systems that arise from analysis in the previous sections are discussed.

5.2 Conjecture for absolute stability

In this section we present a technique to approximate the stability for Lur'e systems that was suggested independently by Shorten [Sho96] and Power & Tsoi [PT74]. The method considers the existence of periodic motion $y(\cdot)$ of the Lur'e system (5.2) for a class of periodic feedback modulations $\phi(\cdot)$. From the non-existence of such periodic motion, absolute stability of the feedback system is suggested. The approach is similar to the describing function approach.[1] The main difference is that here time-varying feedback gains are examined whereas the original approach considers stationary nonlinearities.

In order to highlight the fundamental ideas of the approach, we begin our discussion by considering the feedback gain

$$\phi(t) = -2qcos(\omega_s t), \qquad \omega_s \in \mathbb{R}\,. \tag{5.3}$$

To illustrate the intuition behind the approach we consider the open loop system where the loop is cut open at the output as depicted in Figure 5.2. We assume a periodic signal of the form $y'(t) = 2\cos(\omega_y t + \varphi)$ at the output and follow the signal traversing the loop.

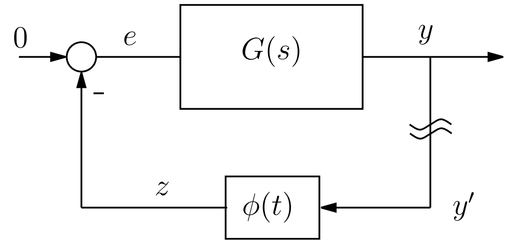


Figure 5.2: Open feedback-loop of the Lur'e system.

[1]The describing function approach is a classical analysis method for stationary feedback nonlinearities. The output of the nonlinearity is approximated by a first-harmonic representation. The analysis aims to detect whether such oscillation is attenuated in the loop. For details see e.g. [Moh91].

Using the exponential description of cosines

$$\begin{aligned} y'(t) &= e^{j(\omega_y t+\varphi)} + e^{-j(\omega_y t+\varphi)} \\ \phi(t) &= -qe^{j\omega_s t} - qe^{-j\omega_s t} \end{aligned}$$

we obtain for $z(t)$ after multiplication

$$-z(t) = qe^{j(\omega_s+\omega_y)t+j\varphi} + qe^{j(-\omega_s+\omega_y)t+j\varphi} + qe^{j(\omega_s-\omega_y)t-j\varphi} + qe^{j(-\omega_s-\omega_y)t-j\varphi}\,.$$

For the next step we introduce the following notation for the gain and argument of the transfer function $G(j\omega)$ at the frequencies of interest:

$$\begin{aligned} g_3 &= \big|G(j(\omega_s+\omega_y))\big|, & \vartheta_3 &= \arg\big\{G(j(\omega_s+\omega_y))\big\}\,, \\ g_1 &= \big|G(j(\omega_s-\omega_y))\big|, & \vartheta_1 &= \arg\big\{G(j(\omega_s-\omega_y))\big\}\,. \end{aligned}$$

For the considered real rational transfer functions holds $\big|G(j\omega)\big| = \big|G(-j\omega)\big|$ and $\arg\{G(j\omega)\} = -\arg\{G(-j\omega)\}$. Substitution yields

$$\begin{aligned} y(t) &= q\,g_3\Big(e^{j(\omega_s+\omega_y)t+j(\varphi+\vartheta_3)} + e^{j(-\omega_s-\omega_y)t-j(\varphi+\vartheta_3)}\Big) + \\ &\quad + q\,g_1\Big(e^{j(-\omega_s+\omega_y)t+j(\varphi-\vartheta_1)} + e^{j(\omega_s-\omega_y)t-j(\varphi-\vartheta_1)}\Big)\,. \end{aligned}$$

Assume now that $G(j\omega)$ has sufficient low-pass characteristic such that $g_3 \ll g_1$. Then $y(t)$ can be approximated by

$$y(t) \approx q\,g_1\left(e^{j(\omega_s-\omega_y)t-j(\varphi-\vartheta_1)} + e^{-j(\omega_s-\omega_y)t+j(\varphi-\vartheta_1)}\right)\,. \tag{5.4}$$

Subject to this approximation, the signal $y'(t)$ is reproduced at the output only if $\omega_y = \frac{1}{2}\omega_s$, $\varphi_0 = \frac{1}{2}\vartheta_1$ and the gain of the transfer function satisfies $\big|\,qG(j\frac{\omega_s}{2})\big| = 1$.

Assuming that the absence of periodic motion is sufficient for stability, the Lur'e system is suggested to be stable [Sho96] for the time-varying feedback gain (5.3) if

$$\big|qG(j\omega)\big| < 1 \qquad \forall\,\omega\in\mathbb{R}\,. \tag{5.5}$$

In [PT74] this technique is applied to analyse the feedback system (5.2) for a class of piecewise constant periodic feedback gains. Consider the piecewise constant function $\phi:\mathbb{R}\to\{k_1,k_2\}$ of period $\frac{T}{2}$ in Figure 5.3 with exactly two discontinuities per period. Let $t=t_0$ be the beginning of one such period then $\phi(\cdot)$ is defined by

$$\phi(t) = \begin{cases} k_1 & \text{for} \quad t_0 \le t < t_0+\Delta\frac{T}{2} \\ k_2 & \text{for} \quad t_0+\Delta\frac{T}{2} \le t < t_0+\frac{T}{2} \end{cases} \tag{5.6}$$

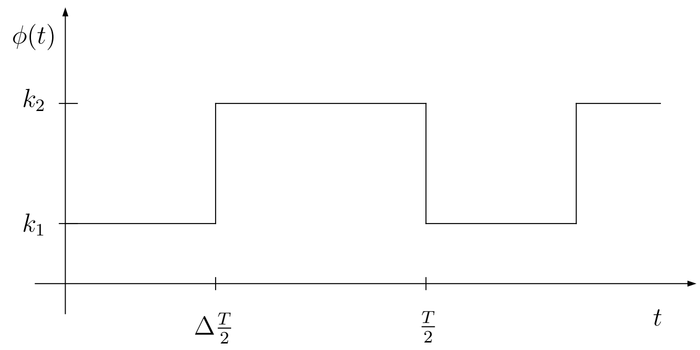


Figure 5.3: Class of nonlinearities considered in [PT74].

where $T > 0\,, k_1, k_2 > 0$, and $\Delta \in [0,1]$ is called the duty-cycle and denotes the ratio of one period for which $\phi(t)$ equals k_1.

In order to apply the ideas of the previous reasoning we approximate the feedback modulation $\phi(\cdot)$ with its first order Fourier expansion

$$\phi(t) \quad \approx \quad \nu_0(\Delta) + 2\nu_1(\Delta)\sin\Big(2\omega(t-t_0) - \frac{\pi}{2}\Big) \tag{5.7}$$

where $\omega = \frac{4\pi}{T}$ and the Fourier coefficients are given by $\nu_0(\Delta) = k_1 + (k_2 - k_1)\Delta$ and $\nu_1(\Delta) = \frac{k_2 - k_1}{\pi}\sin\pi\Delta$.

Neglecting higher-order harmonics, the same calculation as above yields that the periodic motion

$$y(t) \quad \approx \quad \sin\omega t \tag{5.8}$$

can be sustained by the feedback system if

$$\frac{-1}{G(j\omega)} \quad = \quad \nu_0(\Delta) + \nu_1(\Delta)e^{-j\omega t_0} \tag{5.9}$$

for some $\omega > 0$ and $t_0 \in \mathbb{R}$, [PT74].

Condition (5.9) has a simple graphical interpretation. In a manner similar to the describing function approach, the negative inverse Nyquist plot $\frac{-1}{G(j\omega)}$ is compared to the describing function-like representation of the feedback modulation $\phi(\cdot)$. The right-hand side of condition (5.9) describes a family of circles in the complex plane, parameterised by the duty-cycle Δ. For every $\Delta \in [0,1]$ we obtain a circle with radius $\nu_1(\Delta)$ centred at the point $\big(\nu_0(\Delta), 0\big)$. If the inverse Nyquist plot does not intersect any circle of that family, Equation (5.9) does not hold for any pair (Δ, ω). This relation is illustrated by the following example.

Example 5.1

Consider the Lur'e system with linear part

$$G(s) = \frac{-1.1s - 0.7}{s^2 + 2.3s + 1.8}$$

and feedback modulation $\phi(\cdot)$ *of the class* (5.6) *belonging to in the sector* $[1, 2]$.

Figure 5.4 shows the inverse Nyquist plot $\frac{-1}{G(j\omega)}$ *and the circles given in* (5.9) *for some* $\Delta \in [0, 1]$. *Since the inverse Nyquist plot does not intersect any of the circles, we would expect – within some approximation error – that the system does not sustain a periodic motion* $y(t)$ *for any modulation frequency* ω. □

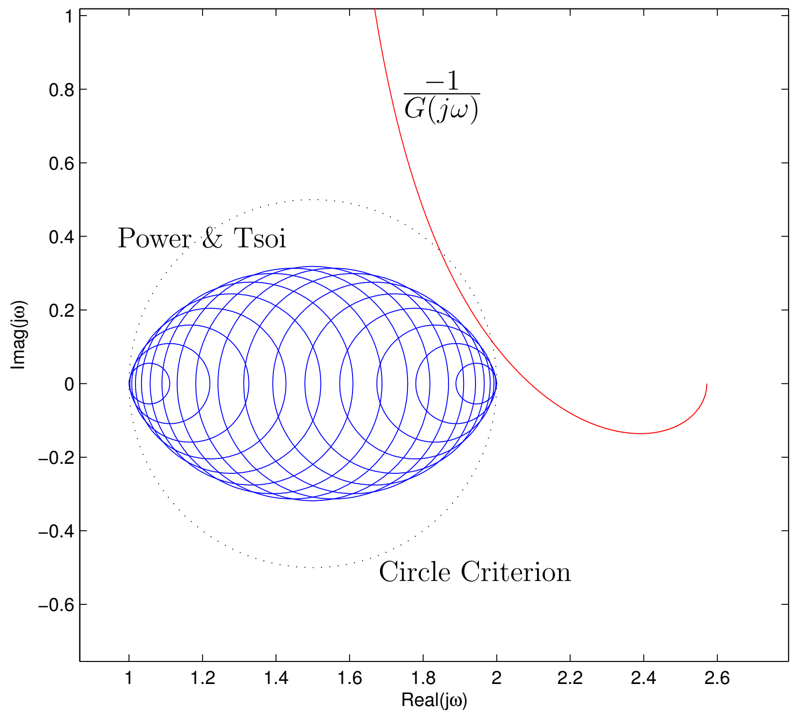


Figure 5.4: Graphical interpretation of the stability conjecture in [PT74]. The family of circles in solid line represents the first-order approximation (5.7) of the feedback modulation. Since the inverse Nyquist plot does not intersect any circle of this family, the system is deemed to not having a periodic solution [PT74]. For comparison, the Circle Criterion requires that $\frac{-1}{G(j\omega)}$ does not intersect the dotted circle.

Note that the above reasoning is based on the considerations of the specific class of time-varying feedback gains (5.6). The authors obtain this class of signals by constructing a feedback modulation that is assumed to be most destabilising for a second-order system. Therefore and upon further observations and empirical studies the following conjecture is formulated:

Conjecture 5.1 *The Lur'e system* (5.2) *is absolutely stable, if the inverse Nyquist plot* $\frac{-1}{G(j\omega)}$ *lies completely outside the family of circles described by* $\nu_0(\Delta)+\nu_1(\Delta)e^{-j\omega t_0}$ *for all* $\omega, t_0 \in \mathbb{R}$ *and* $\Delta \in [0,1]$.

Conjecture 5.1 relaxes the stability condition given by the Circle Criterion, which requires that the inverse Nyquist plot $\frac{-1}{G(j\omega)}$ lies outside the circle $\frac{k_2+k_1}{2} + \frac{k_2-k_1}{2}e^{j\omega t_0}$ (shown as a dotted line in Figure 5.4). According to Conjecture 5.1 the Lur'e system in our example is absolutely stable while the condition of the Circle Criterion is violated and thus, no CQLF exists. If proven true, Conjecture 5.1 would establish stability for a large class of systems for which the Circle Criterion fails.

Unfortunately, as we shall see, the conjecture is not true in general. This is ultimately due to the error caused by the approximations (5.7) and (5.8). However, the examination of the conjecture raises a number of questions which lead to further insights and results.

5.3 Examination of the stability conjecture

In this section we examine the validity of Conjecture 5.1. Although it turns out to be in general not true, Conjecture 5.1 raises a number fundamental theoretical questions which we shall clarify and discuss in this section. In particular the following problems need to be resolved to verify the validity of Conjecture 5.1:

(i) The analysis that leads to Conjecture 5.1 aims to detect periodic motion $y(\cdot)$. It needs to be shown that the non-existence of such periodic motion is sufficient for absolute stability, i.e. that the system is asymptotically stable for all nonlinearities in $[k_1, k_2]$ if no periodic feedback gain $\phi(\cdot)$ belonging to that sector sustains a periodic motion $y(\cdot)$.

(ii) The class of feedback modulations $\phi(\cdot)$ in (5.6) does not represent all periodic signals belonging to $[k_1, k_2]$. Therefore, Conjecture 5.1 implies that the class of signals (5.6) represents (in some sense) the most destabilising periodic feedback modulation belonging to the sector $[k_1, k_2]$.

(iii) The impact of the approximations (5.7) and (5.8) needs to be analysed. For the application of Conjecture 5.1 it is of great importance whether condition (5.9) is sufficient for stability or necessary for instability.

(iv) The authors observe that $y(t)$ is an inverse-repeat wave form of *twice* the period of $\phi(t)$ and base their approach upon this observation.

The above issues are fundamental for the validity and applicability of Conjecture 5.1. In this section we analyse the first three questions and discuss their implications for the stability of switched systems. The last point is briefly addressed in Section 5.4.1.

5.3.1 On periodic and asymptotic stability

In this section we shall investigate whether the absence of periodic motion for all periodic feedback modulations belonging to the sector $[0, k]$ is sufficient for the absolute stability of the Lur'e system with nonlinearities belonging to this sector. By Theorem 4.2 the absolute stability of the Lur'e system is equivalent to the asymptotic stability of the corresponding switched system. Since the switched system is obtained by piecewise constant feedback modulations taking on values on the boundary of the sector, we shall only consider this class of signals for the remainder of this discussion.

More precisely, we consider the following problem: Let $\mathcal{S}(k)$ denote the set of piecewise constant functions $\phi(\cdot) : \mathbb{R} \to \{0, k\}$ and let $\mathcal{S}_p(k) \subset \mathcal{S}(k)$ be the set of periodic functions in $\mathcal{S}(k)$. We call the Lur'e system (5.2) with nonlinearities belonging to sector $[0, k]$ *periodically* stable if it is asymptotically stable for all $\phi \in \mathcal{S}_p(k)$. In the following we investigate whether periodic stability of (5.2) implies its absolute stability.

Consider the state-transition matrix $\Phi_\phi(t, t_0)$ which yields the solution of the Lur'e system (5.2) for a given feedback modulation $\phi(\cdot) \in \mathcal{S}$

$$x(t, t_0, x_0, \phi) \quad = \quad \Phi_\phi(t, t_0) x(t_0) \, .$$

As in [PR91b] we define a generalised notion of the spectral radius[2]

$$\Re\big(G(s), \mathcal{S}\big) \quad = \quad \limsup_{t\to\infty} \max_{\phi(\cdot)\in\mathcal{S}} \varrho\big(\Phi_\phi(t, t_0)\big) \, .$$

This quantity is similar to generalised spectral radius introduced in [DL92] for discrete-time systems. Simply speaking, $\Re\big(G(s), \mathcal{S}\big)$ denotes the spectral radius of the transition matrix $\Phi_\phi(t, t_0)$ of system (5.2) for the "worst-case" feedback modulation

[2]The spectral radius of the matrix A is defined by $\varrho(A) = \max_{\lambda_i} |\lambda_i|$ where λ_i are the eigenvalues of A.

$\phi(\cdot) \in \mathcal{S}$ for $t \to \infty$. The quantity $\Re\big(G(s), \mathcal{S}\big)$ can only take on three different values [PR91b] which yield immediately the following stability properties for the Lur'e system (5.2):

$$\Re\big(G(s), \mathcal{S}\big) \quad = \quad \begin{cases} 0 & : \quad \text{absolutely stable,} \\ 1 & : \quad \text{marginally stable,} \\ \infty & : \quad \text{unstable.} \end{cases} \tag{5.10}$$

Existence of periodic motion and its relation to absolute stability

The existence of periodic motion in relation to the absolute stability of the Lur'e system (5.2) is investigated in a series of publications by Pyatnitskii and Rapoport. Clearly, if there exists periodic motion $y(\cdot)$ for some $\phi(\cdot) \in \mathcal{S}$ then the Lur'e system is not absolutely stable. The converse problem is considered in [Pya71] for second-order systems, in [PR91a] for third-order systems and for systems of arbitrary order in [PR91b] and [PR96].

Let $k^* < \infty$ denote the least upper bound of k for which the system (5.2) is absolutely stable in $\mathcal{S}(k)$. Under mild conditions Pyatnitskii and Rapoport show the following [PR91b]:

Theorem 5.2 *Consider the system in Figure 5.1. Let $k' > k^*$. Then, there exist $T > 0$, $k \in [k^*, k')$ and a T-periodic function $\phi(\cdot) \in \mathcal{S}(k)$ such that the solution of* (5.2) *is T-periodic.*

In other words, arbitrarily close to (but above) the boundary of stability k^* there exists a *periodic* feedback gain $\phi_j \in \mathcal{S}(k)$ of period T such that $\varrho(\Phi_{\phi_j}(T, 0)) = 1$. Hence, the existence of a periodic solution is a necessary condition for the instability of the system (5.2). Moreover, for second and third order systems it has been established that periodic motion can be found for $\phi(\cdot) \in \mathcal{S}_p(k^*)$, see [Pya71] and [PR91a] or [Bar93], respectively.

Recall that the Lur'e system (5.2) with nonlinearity $\phi(\cdot, \cdot)$ belonging to the sector $[0, k]$ is absolutely stable if and only if is asymptotically stable for all $\phi : \mathbb{R} \to \{0, k\}$. Hence, the Lur'e system of order $n \leq 3$ is absolutely stable if and only if it does not support periodic motion for any $\phi \in \mathcal{S}_p(k)$. For higher order systems the same can be shown under the additional assumption that there exists an invariant cone for the system (5.2). For a proof of this fact see [PR91b] and [PR96].

Sufficient condition for stability

Although algebraic evidence for the existence of periodic motion on the stability boundary k^* for Lur'e systems of order $n > 3$ has yet to be found, the absence of periodic motion for all nonlinearities $\phi \in \mathcal{S}$ is still of some significance for absolute stability [WFS03].

Consider the periodic switching signal $\phi_p(\cdot) \in \mathcal{S}_p$ with period T. From Floquet Theory it can be shown [Rug96] that the system (5.2) is asymptotically stable for the feedback signal $\phi_p(\cdot)$ if and only if the spectral radius of the transition matrix for one period is less than one, i.e.

$$\varrho\big(\Phi_{\phi_p}(T,0)\big) < 1. \tag{5.11}$$

Let $\mathcal{S}_{p,T}$ be the set of T-periodic functions $\phi_{p,T}(\cdot)$. Consider

$$\Re\big(G(s), \mathcal{S}_{p,T}\big) \quad = \quad \max_{\phi_{p,T}(\cdot)\in\mathcal{S}_{p,T}} \varrho\big(\Phi_{\phi_{p,T}}(T,0)\big).$$

Then $\Re\big(G(s), \mathcal{S}_{p,T}\big)$ tends to $\Re\big(G(s), \mathcal{S}\big)$ when $T \to \infty$. Since $\Re\big(G(s), \mathcal{S}_{p,T}\big)$ is continuous in T we obtain a sufficient condition for absolute stability by introducing a notion of robustness $\varepsilon > 0$ that bounds $\Re\big(G(s), \mathcal{S}_{p,T}\big) < 1 - \varepsilon$ for all $T > 0$.

Corollary 5.3 *[WFS03] System* (5.2) *with nonlinearity* $\phi(\cdot,\cdot)$ *belonging to the sector* $[0, k]$ *is absolutely stable, if there exists* $\varepsilon > 0$ *such that* $\varrho(\Phi_{\phi_p}(T,0)) < 1 - \varepsilon$ *for all* T*-periodic functions* $\phi_p(\cdot) \in \mathcal{S}_p(k)$.

5.3.2 Feedback modulation with two switches per period

The previous discussion showed that we can obtain sufficient conditions for absolute stability by considering periodic piecewise constant feedback modulations that only takes on values on the boundary of the sector. However, the reasoning that led to the Conjecture 5.1 only considers a subclass of such periodic functions. The class of functions (5.6) is essentially characterised by having exactly two discontinuities per period; one at $t = \Delta\frac{T}{2}$ and another right at the end of the period $t = \frac{T}{2}$. The authors obtain this system class by constructing a feedback signal that maximises the trajectory slope of a second order system and therefore is deemed be the most destabilising [PT74]. In this section we shall justify the restriction of the condition in Conjecture 5.1 to this class of functions.

The problem whether the Lur'e system is absolutely stable if and only if it is asymptotically stable for all periodic feedback modulations with *two discontinuities* per period has been studied in several publications in the past. In fact, this hypothesis has been proven true for second-order systems [Pya71] and for third-order systems in [Bar93, Rap94]. However, it is unknown whether such assertion is true for systems of order greater than three.

In this section we shall show that the graphical condition of Conjecture 5.1 holds for any piecewise constant periodic modulation signal $\phi : \mathbb{R} \rightarrow \{k_1, k_2\}$ if and only if it holds for the class of periodic modulation signals (5.6) with two switches per period. Geometrically speaking, we show that the area covered by the family of circles obtained by any signal with $2N$ discontinuities per period is included in the area covered by the circles of the signals with two discontinuities (see Figure 5.5).

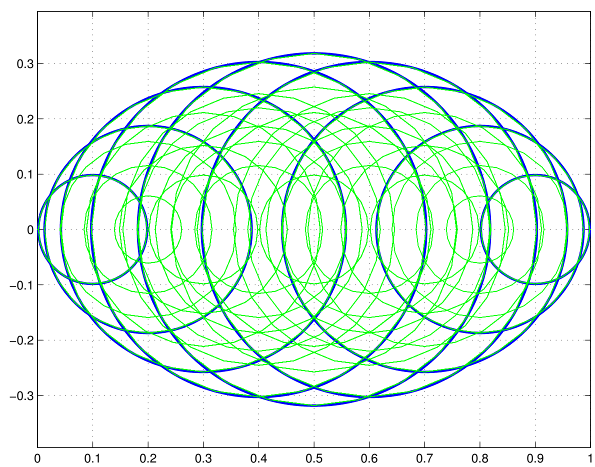


Figure 5.5: Families of circles of the first-order approximation of feedback modulations with 2 discontinuities per period (fat, dark lines), and 4 discontinuities per period (thinner, light gray lines).

The circle associated with the periodic piecewise constant function $\phi(\cdot)$ is given by

$$\nu_0(\phi) + \nu_1(\phi)e^{j\omega t_0} \tag{5.12}$$

where $\nu_0(\phi), \nu_1(\phi)$ are the first two coefficients of the Fourier series of $\phi(\cdot)$. The circles are centred at $\nu_0(\phi)$ and have a radius of $\nu_1(\phi)$. We shall show, that for any given ν_0 the greatest radius $\nu_1(\phi)$ is obtained by a function with two discontinuities per period.

Consider the class $\mathcal{V}(\nu_0)$ of periodic functions $\phi(\cdot)$ with the average value ν_0 that consist of square pulses of equal magnitude (as depicted in Figure 5.6). Without loss of generality we assume that magnitude and period are normalised to unity. The function $\phi_N(\cdot) \in \mathcal{V}(\nu_0)$ consists of $N \in \mathbb{N}$ pulses, where pulse i starts at time t_i and has the length $\tau_i > 0$. Since all functions in $\mathcal{V}(\nu_0)$ have the same average value, it follows that $\sum_i^N \tau_i = \nu_0$.

Let $\nu_1(\phi_N)$ denote the Fourier coefficient of the first harmonic of ϕ_N. We show that the largest magnitude of the first harmonic

$$\hat{\nu}_1(\nu_0) \quad = \max_{N\in\mathbb{N},\phi_N\in\mathcal{V}(\nu_0)} \left|\nu_1(\phi_N)\right|$$

is attained by the function $\phi_N(t)$ with $N = 1$ for all $\nu_0 \in (0, 1)$.

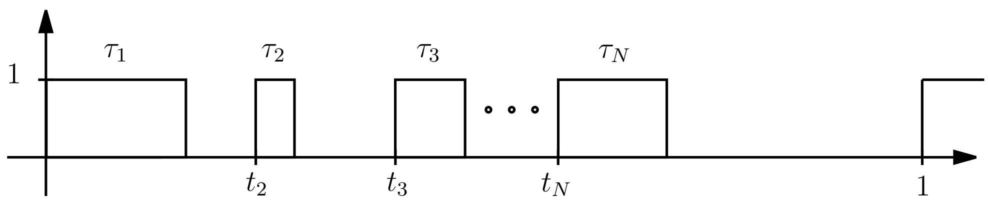


Figure 5.6: Feedback signal $\phi_N(t)$.

Note, that a phase shift of the function only affects the phase of the first harmonic but not its magnitude. Therefore we assume without loss of generality that the first pulse of $\phi(\cdot) \in \mathcal{V}(\nu_0)$ starts at $t = 0$ and the last pulse ends at $t_N + \tau_N < T = 1$.

The idea of the proof is the following. Consider the function $\phi_{N+1} \in \mathcal{V}(\nu_0)$

$$\phi_{N+1}(t) = \phi_1(t) + \phi_N(t - \varphi) \tag{5.13}$$

where $\phi_1(t) \in \mathcal{V}(\tau_0)$ and $\phi_N(t) \in \mathcal{V}(\nu_0 - \tau_0)$ as depicted in Figure 5.7. ϕ_1 is shown as a solid line and ϕ_N is shown as a dashed line with the time-shifts *varphi* taking on different values for each sub-figure. Since we require that $\phi_{N+1} \in \mathcal{V}(\nu_0)$, the time-shift φ is bounded by $\varphi \in [\tau_0, 1 - (t_N + \tau_N)]$. For the same reason we need to restrict $t_N + \tau_N < 1 - \tau_0$. We shall show that the magnitude of the first harmonic $\left|\nu_1(\phi_{N+1}(t))\right|$ of the function $\phi_{N+1}(t)$ parameterised in φ is maximal if $\varphi = \tau_0$ (Figure 5.7b) or $\varphi = 1 - (t_N + \tau_N)$ (Figure 5.7c). In these two cases the function in (5.13) essentially only consists of N pulses. We can now use the same arguments to shift the remaining $N - 1$ pulses. By induction $\hat{\nu}_1(\nu_0)$ is attained if all pulses are shifted to one side, which is equivalent to $\phi_1(t) \in \mathcal{V}(\nu_0)$.

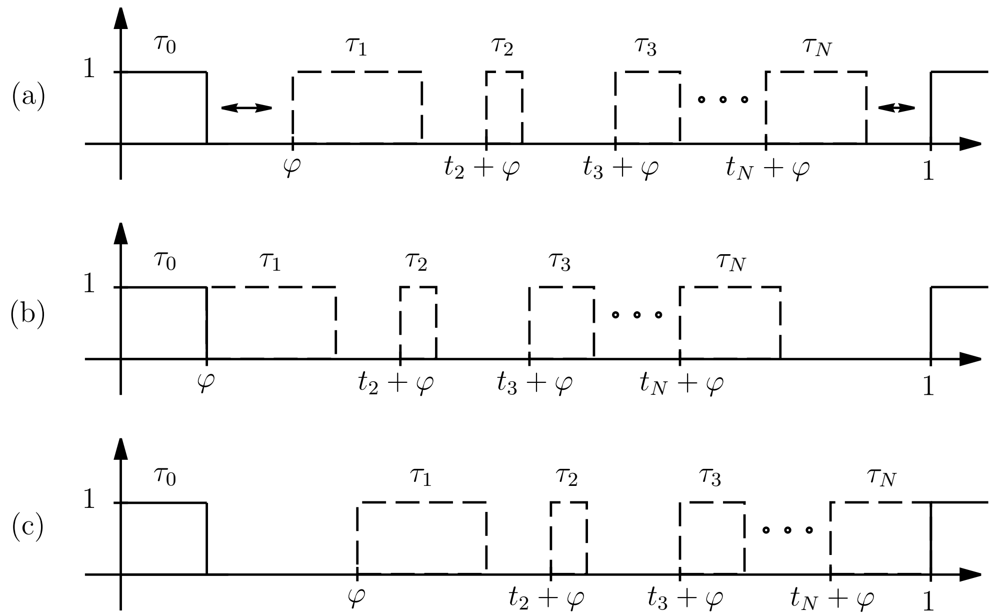


Figure 5.7: Construction of the function $\phi_{N+1}(t)$ in Equation (5.13) for different values of φ. (a) $\tau_0 < \varphi < 1-(t_N+\tau_N)$; (b) $\varphi = \tau_0$; (c) $\varphi = 1-(t_N+\tau_N)$.

The Fourier coefficient of the first harmonic of ϕ_N is given by

$$\begin{aligned}
\nu_1(\phi_N) &= \frac{1}{T}\int_1^T f(t)e^{-j\omega t}dt \\
&= \frac{1}{j2\pi}\sum_{i=1}^{N} e^{-j2\pi t_i} - e^{-j2\pi(t_i+\tau_i)} \\
&= \frac{1}{\pi}\sum_{i=1}^{N} \sin(-\pi\tau_i)\, e^{-j\pi(2t_i+\tau_i)}
\end{aligned}$$

where $t_i \in [0, 1-\tau_N-\tau_0)$. We can consider $\nu_1(\phi_N)$ as a sum of vectors with magnitude $|\sin(\pi\tau_i)|$. Let $-\theta$ be the argument of $\nu_1(\phi_N)$. Neglecting some multiple of 2π the argument θ is bounded by the smallest and largest angle in the sum, i.e. $\theta \in [\tau_1, 2t_N + \tau_N)$.

Using the above construction of the function ϕ_{N+1} in (5.13) we get the Fourier coefficient of its first harmonic

$$\nu_1(\phi_{N+1}) = \frac{1}{\pi}\sin(-\pi\tau_0)\, e^{-j\pi\tau_0} + e^{-j2\pi\varphi}\nu_1(\phi_N)$$

where the time-shift $\varphi \in [\,\tau_0,\, 1-(t_N+\tau_N)\,]$.

Substituting $\nu_1(\phi_N) = |\nu_1(\phi_N)|e^{-j\theta}$ yields

$$\nu_1(\phi_{N+1}) \quad = \quad \frac{1}{\pi}\sin(-\pi\tau_0)e^{-j\pi\tau_0} + \big|\nu_1(\phi_N)\big|e^{-j\pi(2\varphi+\theta)} \tag{5.14}$$

Consider $\nu_1(\phi_{N+1})$ as a sum of two vectors. Then $|\nu_1(\phi_{N+1})|$ is maximal if the angle between the two vectors is minimal, i.e.

$$\max_{\varphi} |\nu_1| \Longleftrightarrow \min_{\varphi} |\, \pi(\tau_0 - 2\varphi - \theta)\, |$$

We shall now show that the minimum of $\pi(\tau_0 - 2\varphi - \theta)$ is obtained for φ at the boundary of the interval $[\,\tau_0,\, 1 - (t_N + \tau_N)\,]$.

The function $f(\varphi) = |\pi(\tau_0 - 2\varphi - \theta)\,|$ has the minima $\varphi_{0,n}$ for which

$$|\,\tau_0 - 2\varphi_{0,n} - \theta\,| = 2n, \qquad n \in \mathbb{Z}$$

We shall show that $\varphi \in \big[\tau_0,\, 1 - (t_N + \tau_N)\big]$ lies between the two consecutive minima $\varphi_{0,0}$ and $\varphi_{0,1}$ for $n = 0$ and $n = 1$, respectively.

For $n = 0$ we obtain for the minimum $\varphi_{0,1} = \frac{1}{2}(\tau_0 - \theta)$. Hence we require

$$\begin{aligned} \frac{1}{2}(\tau_0 - \theta) \quad &< \quad \tau_0 \\ -\theta \quad &< \quad \tau_0 \end{aligned}$$

Since $\theta \geq \tau_1 > 0$ this inequality holds and $\varphi \in \big[\tau_0,\, 1 - (t_N + \tau_N)\big]$ is always greater than $\varphi_{0,0}$.

The minimum for $n = 1$ is given by $\varphi_{0,1} = 1 + \frac{1}{2}(\tau_0 - \theta)$. We require that $\varphi_{0,1}$ is greater than the upper bound $1 - (t_N + \tau_N)$, i.e.

$$\begin{aligned} 1 + \frac{1}{2}(\tau_0 - \theta) \quad &> \quad 1 - (t_N + \tau_N) \\ \tau_0 - \theta \quad &> \quad -2t_N - 2\tau_N \\ \tau_0 \quad &> \quad -2t_N - 2\tau_N + \theta \end{aligned}$$

Substituting the upper bound for $\theta = 2t_N + \tau_N$ gives

$$\tau_0 \quad > \quad -\tau_N$$

Since $\tau_N > 0$ we can conclude that $\varphi \in [\,\tau_0,\, 1 - (t_3 + \tau_3)\,]$ lies between consecutive minima. That implies that $|\nu_1(\phi_{N+1})|$ is maximal for φ at the boundaries of that interval. It follows by induction that the maximum $\hat{\nu}_1(\phi_N)$ for $\phi_N \in \mathcal{V}(\nu_0)$ is attained for the function $\phi_1(t) \in \mathcal{V}(\nu_0)$ for all $\nu_0 \in (0, 1)$.

We conclude, that the condition of Conjecture 5.1 holds for all periodic piecewise constant feedback modulations if it holds for the considered class of functions (5.6).

Further recall that in [Pya71, Bar93, Rap94] is shown that the class of periodic, piecewise constant feedback modulations with two discontinuities per period is necessary and sufficient for determining absolute stability of Lur'e system of order $n \leq 3$. In the sense of a first-order approximation of the feedback modulation ϕ the above findings generalise this result for Lur'e systems of arbitrary order. While the derivation in this section does not allow any conclusions for absolute stability, it encourages to conjecture that the results of [Pya71, Bar93, Rap94] could be generalised for Lur'e systems of arbitrary order.

5.3.3 Evaluation of the approximation error

In the previous two sections the fundamental assumptions that lead to the Conjecture 5.1 have been discussed and justified. In this section we shall see that the error due to the approximations (5.7) and (5.8) is not systematic in the sense that the condition of Conjecture 5.1 is neither sufficient for the existence of periodic motion nor sufficient for its non-existence.

The fact that Conjecture 5.1 only considers feedback modulations of the form (5.6), allows the direct verification by computing the transition matrix $\Phi_\sigma(T,0)$ for one period for the respective values of Δ and T and applying condition (5.11) obtained by Floquet Theory [Rug96]. The Lur'e system (5.2) is asymptotically stable for the periodic feedback modulation $\phi(\cdot)$ in (5.6) if and only if

$$\varrho\left(e^{\left(A+kbc^{\mathsf{T}}\right)(1-\Delta)T}\, e^{\left(A-kbc^{\mathsf{T}}\right)\Delta T}\right) < 1\,. \tag{5.15}$$

Therefore, we can evaluate the approximation error directly and independently of the discussion of the previous sections.

In the following we present two examples to demonstrate that Conjecture 5.1 is neither sufficient for stability nor sufficient for instability.

Example 5.2 *Consider the switched linear system* (5.1) *with*

$$A_1 = \begin{pmatrix} 0 & 1 \\ -0.5 & -2.5 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -190.5 & -36.0 \end{pmatrix}.$$

The corresponding Lur'e system (5.2) *is given by the transfer function*

$$G(s) \quad = \quad \frac{-16.75s - 95}{s^2 + 19.25s + 95.5}$$

and the nonlinearities $\phi(\cdot,\cdot)$ *belonging to the sector* $[-1, 1]$.

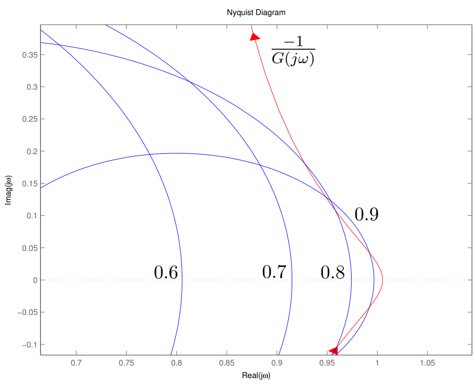


Figure 5.8: Graph of Conjecture 5.1 for Example 5.2.

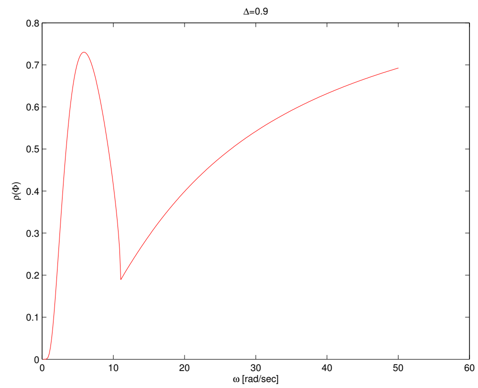


Figure 5.9: Spectral radius of the transition matrix $\Phi\left(\frac{2\pi}{\omega}, 0\right)$ for $\Delta = 0.9$.

Figure 5.8 shows the negative inverse Nyquist plot of $G(j\omega)$ *and circle segments for some* $\Delta \in [0, 1]$ *corresponding to the first-order approximation of the signal* (5.6). *The inverse Nyquist plot intersects some of these circles. In particular, we can read off intersections for* $\Delta = 0.9$ *at frequencies* $\omega_0 \approx 1.5\frac{\text{rad}}{\text{s}}$ *and* $\omega_0 \approx 3.65\frac{\text{rad}}{\text{s}}$ *of* $\frac{-1}{G(j\omega)}$. *According to Conjecture 5.1 we would expect a periodic output-signal* $y(t)$ *of frequency* ω_0 *if* $\phi(t)$ *switches with a duty-cycle* $\Delta = 0.9$ *and frequency* $2\omega_0$.

We verify this finding by calculating the spectral radius (5.15) *for the duty-cycle* $\Delta = 0.9$ *shown in Figure 5.9. The graph does not exceed* 0.8 *for the frequencies in question. Hence, the system does not sustain any periodic motion for that duty-cycle. Further evaluation reveals that the spectral radius does not exceed* 1 *for any duty-cycle* $\Delta \in [0, 1]$ *and any modulation frequency. Note that the spectral radius converges to* 1 *for* $\omega \to \infty$ *since*

$$\lim_{\omega\to\infty} e^{A\frac{2\pi}{\omega}} \quad = \quad I\,.$$

Hence, Conjecture 5.1 is not sufficient for the existence of periodic motion. □

Example 5.3 *Consider the switched linear system* (5.1) *with*

$$A_1 \quad = \quad \begin{pmatrix} 0 & 1.0 \\ -0.0250 & -0.6050 \end{pmatrix}, \qquad A_2 \quad = \quad \begin{pmatrix} 0 & 1.0 \\ -0.7430 & -0.2610 \end{pmatrix}.$$

The corresponding Lur'e system is given by the transfer function

$$G(s) = \frac{0.172s - 0.359}{s^2 + 0.433s + 0.384}$$

and nonlinearity $\phi(\cdot,\cdot)$ belonging to the sector $[-1, 1]$.

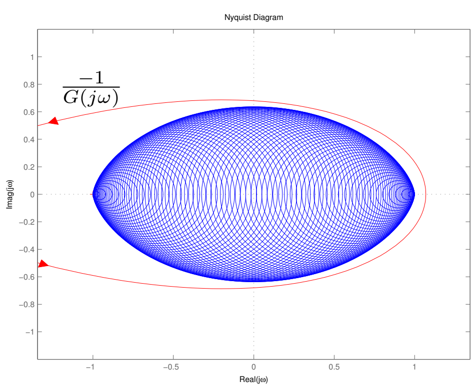


Figure 5.10: Graph of Conjecture 5.1 for Example 5.3.

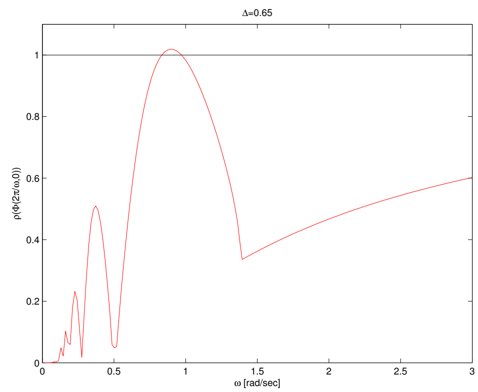


Figure 5.11: Spectral radius of the transition matrix $\Phi\left(\frac{2\pi}{\omega}, 0\right)$ for $\Delta = 0.65$.

The negative inverse Nyquist plot of $G(s)$ is shown in Figure 5.10 together with the set of circles given by the first-order approximation of $\phi(t)$ in (5.6). The Nyquist plot does not intersect any of the circles. Hence, by Conjecture 5.1 we would not expect any periodic solution for the Lur'e system (5.2) and the system is deemed to be absolutely stable.

The spectral radius of the transition matrix $\Phi(T, 0)$ for $\Delta = 0.65$ is shown in Figure 5.11. For $T = \frac{2\pi}{0.9}$ we can read off a value of approximately 1.02 for the spectral radius. Hence the system is just unstable for such a switching signal and Conjecture 5.1 is not sufficient for absolute stability. □

The above examples demonstrate that the approximation inherent in (5.9) leads to ambiguous results. Therefore we conclude that the condition of Conjecture 5.1 is neither a sufficient for stability nor a sufficient for instability.

5.4 The boundary of absolute stability

In the previous section we examined the validity of the Conjecture 5.1 and discussed various questions that arise from it. Although several implications of the conjecture have been resolved, we have nevertheless shown that it fails to provide sufficient conditions for absolute stability. This is mainly due to the error caused by the approximation of the periodic signals $y(\cdot)$ and $\phi(\cdot)$ by their first harmonic. In this section we incorporate all harmonics of these periodic signals to derive necessary and sufficient conditions for the existence of periodic motion. By introducing a notion of robustness as in Corollary 5.3 we obtain sufficient conditions for absolute stability of the Lur'e system.

Our approach here is similar to the approaches in Section 5.2. But instead of considering the linear forward part $G(s)$ and the time-varying feedback gain $\phi(t)$ separately, both elements are lumped together into a single operator $\boldsymbol{F}_{\omega_0}$ describing the open loop dynamics (c.f. Figure 5.12 on page 123). This operator $\boldsymbol{F}_{\omega_0} : L_2[0,T_0] \to L_2[0,T_0]$ is defined on the L_2 space of T_0-periodic functions:

$$L_2[0,T_0] \quad = \quad \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; f(t+T_0) = f(t)\,,\; \int_0^{T_0} \left|f(t)\right|^2 dt \;<\; \infty \right\}.$$

A simple necessary and sufficient condition for the existence of a T_0-periodic solution $y(\cdot) \in L_2[0,T_0]$ for the autonomous closed loop system is derived. This result, together with Theorem 5.2 on existence of periodic solutions, and Corollary 5.3, is used to derive a sufficient condition for absolute stability that approximates the stability boundary to an arbitrary degree of accuracy.

5.4.1 Frequency-ratio of the periodic motion and the periodic feedback modulation

Consider the signals $y(t), \phi(t)$ and $z(t)$ of the closed loop Lur'e system depicted in Figure 5.1 and assume that $y(\cdot)$ is a periodic motion sustained by the periodic feedback modulation $\phi(\cdot)$. In the following we investigate how the frequencies $\omega = \frac{2\pi}{T}$ of these two functions relate.

We denote the period of $\phi(\cdot)$ by T_ϕ and the period of $y(\cdot)$ by T_y.[3] Then the respective Fourier series are given by

$$\phi(t) \quad = \quad \sum_{n=-\infty}^{\infty} \nu_n e^{-jn\omega_\phi t} \tag{5.16}$$

$$y(t) \quad = \quad \sum_{k=-\infty}^{\infty} y_k e^{-jk\omega_y t} \tag{5.17}$$

[3]Here, the term "T-periodic" is used in the minimal sense such that $T \in \mathbb{R}^+$ is the smallest value for which $\phi(t+T) = \phi(t)$.

where ν_n and y_k denote the respective Fourier coefficient; $\omega_\phi = \frac{2\pi}{T_\phi}$ and $\omega_y = \frac{2\pi}{T_y}$ are the fundamental harmonics of $\phi(\cdot)$ and $y(\cdot)$, respectively.

For the signal $z(\cdot)$ we obtain after multiplication

$$z(t) \;=\; \sum_k \sum_n \nu_n y_k \, e^{-j(n\omega_\phi + k\omega_y)t} \,. \tag{5.18}$$

For the output $y(\cdot)$ we get

$$y(t) \;=\; \sum_k \sum_n g_{nk} \nu_n y_k \, e^{-j(n\omega_\phi + k\omega_y)t} \tag{5.19}$$

where $g_{nk} = G\big(j(n\omega_\phi + k\omega_y)\big)$.

By assumption, $y(\cdot)$ is a periodic motion sustained by the feedback system (5.2). Thus the description of the output signal $y(\cdot)$ in (5.17) and (5.19) are equivalent. It follows that for any frequency component $n\omega_\phi + k\omega_y$ in (5.19) there is a frequency component $l\omega_y$ in (5.17), i.e. for every $n, k \in \mathbb{Z}$ there exists an $l \in \mathbb{Z}$ such that

$$n\omega_\phi + k\omega_y \;=\; l\omega_y \,. \tag{5.20}$$

Choosing $n = 1$ we get $\omega_\phi = (l - k)\omega_y$, i.e. the fundamental frequency ω_ϕ of the modulation signal $\phi(\cdot)$ is an integer multiple of the frequency of the output signal $y(\cdot)$. However, from (5.20) we cannot deduce that the ratio of the fundamental harmonics $r = \frac{l-k}{n}$ equals two as assumed in the reasoning that led to the Conjecture 5.1.

5.4.2 The open-loop operator

In Section 5.3.1 it is established that the existence of a periodic solution is necessary for the instability of the Lur'e system (5.2), and conversely, the nonexistence of periodic solutions (in a robust sense) can be used to approximate the stability boundary to an arbitrary degree of accuracy (Corollary 5.3). Therefore we shall only consider periodic functions $y(\cdot)$ in this section and check whether the Lur'e system (5.2) can sustain any such solutions for any $\phi(\cdot)$ belonging to the sector $[k_1, k_2]$.

Consider the space $L_2[0, T_0]$ of T_0-periodic functions $f : \mathbb{R} \to \mathbb{R}$. Let the functions $\phi(\cdot), y(\cdot), z(\cdot) \in L_2[0, T_0]$ where T_0 is the smallest $T_0 \in \mathbb{R}^+$ such that $y(t + T_0) = y(t)$. With the Lur'e system (5.2) we associate the operator $\boldsymbol{F}_{\omega_0} : L_2[0, T_0] \to L_2[0, T_0]$ as depicted in Figure 5.12. In the following we investigate conditions for which $\boldsymbol{F}_{\omega_0}(G, \phi)$ has a solution $y(\cdot) \in L_2[0, T_0]$.

Assume that there exists a periodic feedback modulation $\phi(\cdot)$ for which the Lur'e system sustains some periodic motion $y(\cdot) \in L_2[0, T_0]$. In the previous section it was shown that there exists an integer $r > 0$ such that $\omega_\phi = r\omega_y = r\omega_0$. We denote the periodic feedback modulation with frequency $r\omega_0$ by $\phi_r(\cdot) \in L_2[0, T_0]$. Then the Fourier expansions of $y(t)$ and $\phi_r(t)$ with respect to the basis frequency $\omega_0 = \frac{2\pi}{T_0}$ are given by

$$
\begin{aligned}
y(t) &= \sum_k y_k e^{-j\omega_0 t} \\
\phi_r(t) &= \sum_n \nu_n e^{-jnr\omega_0 t} \\
&= \sum_l \tilde{\nu}_l e^{-jl\omega_0 t}
\end{aligned}
$$

where $\tilde{\nu} = \nu_n$ for $l = nr$ and $\tilde{\nu} = 0$ otherwise.

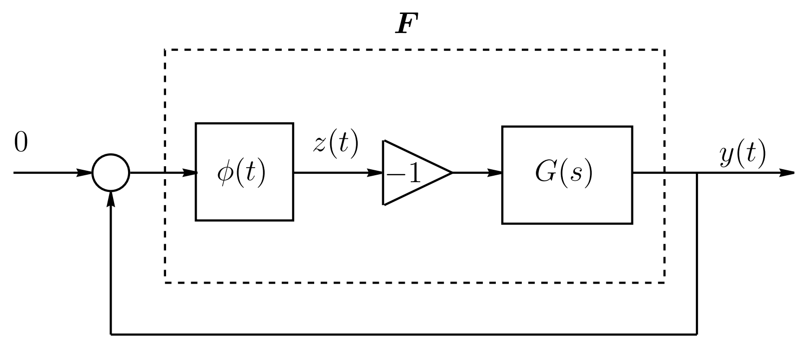


Figure 5.12: Open loop operator $\boldsymbol{F}$.

The functions $y(\cdot)$ and $\phi_r(\cdot)$ can be expressed with respect to the basis $e^{-jk\omega_0 t}, k \in \mathbb{Z}$. We obtain the infinite dimensional vectors $\boldsymbol{y}_{\omega_0}$ and $\boldsymbol{\phi}_{r,\omega_0}$ representing the periodic functions $y(t)$ and $\phi_r(t)$ in this space, respectively; namely,

$$
\begin{aligned}
\boldsymbol{y}_{\omega_0} &= \begin{pmatrix} \cdots \ y_{-r} \ \cdots \ y_{-1} \ y_0 \ y_1 \ \cdots \ y_r \ \cdots \end{pmatrix}^{\mathsf{T}}, \qquad (5.21) \\
\boldsymbol{\phi}_{r,\omega_0} &= \begin{pmatrix} \cdots 0 \ \nu_{-1} \ 0 \ \cdots \ 0 \ \nu_0 \ 0 \ \cdots \ 0 \ \nu_1 \ 0 \cdots \end{pmatrix}^{\mathsf{T}} \qquad (5.22)
\end{aligned}
$$

where the entries in $\boldsymbol{\phi}_{r,\omega_0}$ with consecutive Fourier coefficients of ν_i and ν_{i+1} are separated by $r - 1$ zero-entries.

As in Section 5.2 we shall now follow the signal traversing the loop. In the frequency domain we obtain the spectrum of $z(t)$ by convolution of the spectra of $y(t)$ and $\phi_r(t)$.

Define $\boldsymbol{V}_{\omega_0}(\phi_r)$ as the infinite dimensional Toeplitz matrix constructed by $\boldsymbol{\phi}_{r,\omega_0}$:

$$\boldsymbol{V}_{\omega_0}(\phi_r) = \begin{pmatrix} \ddots & & & & \ddots & & & & \ddots & & & & \\ & \nu_1 & 0 & \cdots & 0 & \nu_0 & 0 & \cdots & 0 & \nu_{-1} & 0 & \cdots & \\ \cdots & 0 & \nu_1 & 0 & \cdots & 0 & \nu_0 & 0 & \cdots & 0 & \nu_{-1} & 0 & \cdots \\ & \cdots & 0 & \nu_1 & 0 & \cdots & 0 & \nu_0 & 0 & \cdots & 0 & \nu_{-1} & \\ & & & & \ddots & & & & \ddots & & & & \ddots \end{pmatrix}.$$

From (5.18) we obtain

$$\boldsymbol{z}_{\omega_0} = \boldsymbol{V}_{\omega_0}(\phi_r)\boldsymbol{y}_{\omega_0}\,.$$

We can now express the components y_k of $\boldsymbol{y}_{\omega_0}$ by the products of z_k and the value of the transfer function $G(s)$ evaluated at the respective frequencies $jk\omega_0$. Define the infinite-dimensional diagonal matrix

$$\boldsymbol{G}_{\omega_0} = \operatorname{diag}\big(\cdots G(-jr\omega_0)\cdots G(-j\omega_0)\ \ G(0)\ \ G(j\omega_0)\ \ \cdots G(jr\omega_0)\cdots\big).$$

Then we obtain

$$\boldsymbol{y}_{\omega_0} = -\boldsymbol{G}_{\omega_0}\boldsymbol{z}_{\omega_0} = -\boldsymbol{G}_{\omega_0}\boldsymbol{V}_{\omega_0}(\phi_r)\boldsymbol{y}_{\omega_0}\,.$$

Define $\boldsymbol{F}_{\omega_0}(G,\phi_r) = -\boldsymbol{G}_{\omega_0}\boldsymbol{V}_{\omega_0}(\phi_r)$. Then

$$\boldsymbol{y}_{\omega_0} = \boldsymbol{F}_{\omega_0}(G,\phi_r)\boldsymbol{y}_{\omega_0}. \tag{5.23}$$

The operator $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ defines the open loop dynamics of the Lur'e system in Figure 5.12 on the space $L_2[0,T_0]$. Note, that $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ depends on the chosen basis $\omega_0 = \frac{2\pi}{T_0}$, the frequency ratio r and the shape of $\phi_r(t)$ over one period which determines the non-zero entries of $\boldsymbol{V}_{\omega_0}(\phi_r)$.

It follows from Equation (5.23) that the periodic motion $\boldsymbol{y}_{\omega_0}$ is an eigenvector of the operator $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ associated with the eigenvalue $\lambda = 1$.

Theorem 5.4 (Necessary and sufficient condition for periodic motion) *The Lur'e system* (5.2) *with the periodic feedback gain* $\phi_r(\cdot) \in L_2[0,T_0]$ *with period* $\frac{T_0}{r}$ *has a* T_0*-periodic solution* $y(\cdot) \in L_2[0,T_0]$ *if and only if* $\boldsymbol{F}_{\omega_0}(G,\phi_r) : L_2[0,T_0] \to L_2[0,T_0]$ *has an eigenvalue equal to one.*

Proof. The theorem follows immediately from the above derivations and in particular from Equation (5.23). □

The above theorem enables us to verify whether the Lur'e system with a given periodic feedback gain $\phi_r(\cdot)$ has a periodic solution $y(\cdot)$ of a certain frequency. Consider now the Lur'e system with feedback modulation $\phi_r(\cdot)$ belonging to the symmetric sector $[-k', k']$. Applying Theorem 5.4 we can identity the smallest sector $[-k'', k'']$, for which there exists a periodic solution $y(\cdot)$.

Theorem 5.5 *Consider the Lur'e system* (5.2) *with* $\phi(\cdot)$ *belonging to* $[-k', k']$. *Let* λ' *be the supremum of the real eigenvalues of the operators* $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ *for all* $\omega_0, r > 0$ *and all periodic* $\phi_r(\cdot) \in L_2[0, T_0]$ *belonging to the sector* $[-k', k']$. *Then there exists a periodic motion* $y(\cdot) \in L_2[0, T_0]$ *for the Lur'e system* (5.2) *with any sector nonlinearities belonging to* $[-k'', k'']$ *where* $k'' > \frac{k'}{\lambda'}$.

Proof. Let λ' be the greatest real eigenvalue of $\boldsymbol{F}_{\omega_0}(G, \phi_r')$ where $\phi_r'(t)$ is some piecewise constant function of period $\frac{T_0}{r}$ belonging to the sector $[-k', k']$. Consider now the function $\phi_r''(t) = \frac{1}{\lambda'}\phi_r'(t)$. The operator $\boldsymbol{F}_{\omega_0}\big(G, \phi_r''\big)$ is given by

$$\boldsymbol{F}_{\omega_0}\big(G, \phi_r''\big) \quad = \quad \boldsymbol{G}_{\omega_0}\boldsymbol{V}_{\omega_0}\big(\phi_r''\big) \quad = \quad \boldsymbol{G}_{\omega_0}\boldsymbol{V}_{\omega_0}\big(\tfrac{1}{\lambda'}\phi_r'\big).$$

Let ν_k'' and the ν_k' denote the Fourier coefficients of $\phi_r''(t)$ and $\phi_r'(t)$, respectively. Since the sector bounds are symmetric $\nu_k'' = \frac{1}{\lambda'}\nu_k'$, $\forall\ k$, thus $\boldsymbol{V}_{\omega_0}\big(\phi_r''\big) = \frac{1}{\lambda'}\boldsymbol{V}_{\omega_0}\big(\phi_r'\big)$. Substitution yields

$$\begin{aligned} \boldsymbol{F}_{\omega_0}\big(G, \phi_r''\big) &= \tfrac{1}{\lambda'}\,\boldsymbol{G}_{\omega_0}\boldsymbol{V}_{\omega_0}\big(\phi_r'\big) \\ &= \tfrac{1}{\lambda'}\boldsymbol{F}_{\omega_0}\big(G, \phi_r'\big). \end{aligned}$$

Hence, $\boldsymbol{F}_{\omega_0}\big(G, \phi_r''\big)$ has an eigenvalue equal to one, and by Theorem 5.4 the feedback modulation $\phi_r''(\cdot)$ sustains a periodic motion $y(\cdot) \in L_2[0, T_0]$.

Let λ' now be the supremum of the real eigenvalues of $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ for all $\omega_0, r > 0$ and all periodic $\phi_r(t) \in L_2[0, T_0]$ belonging to $[-k', k']$. Then there exists a periodic feedback modulation $\phi_r(\cdot)$ belonging to $[-k'', k'']$ with $k'' > \frac{k'}{\lambda'}$ such that $\phi_r(\cdot)$ sustains periodic motion for the Lur'e system. □

For second and third order systems it has been established in [Pya71, Bar93] that there exists periodic motion of the type (5.6) on the stability boundary. Therefore k'' in the previous theorem gives the least upper bound for which the Lur'e system with symmetric sector bounds is absolutely stable.

Corollary 5.6 (Necessary and sufficient condition) *The Lur'e system* (5.2) *with strictly proper transfer function* $G(s)$ *of order two or three and feedback modulation* $\phi(\cdot)$ *belonging to the sector* $[-k,k]$ *is absolutely stable if and only if* $\hat{\lambda} < 1$*, where* $\hat{\lambda}$ *denotes the greatest real eigenvalue of* $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ *for all* $r > 0$ *and all periodic functions* $\phi_r(\cdot)$ *with two discontinuities per period belonging to the sector* $[-k,k]$.

So far it has not been shown that there also exists a periodic solution on the stability boundary k^* for systems of order greater than three. However, there exists periodic motion arbitrarily close, but beyond the stability boundary (Theorem 5.2). Therefore, we can approximate the boundary of absolute stability for the Lur'e system (5.2) as close as we wish by applying a notion of robustness as in Corollary 5.3.

Corollary 5.7 (Maximum sector for absolute stability) *Consider the Lur'e system* (5.2) *with symmetric sector bounds. Let* $k' > 0$ *and let* λ' *be the greatest real eigenvalue of* $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ *for all* $\omega_0, r > 0$ *and all* $\phi_r(\cdot)$ *belonging to the sector* $[-k',k']$*. Then the Lur'e system* (5.2) *is absolutely stable for feedback nonlinearities belonging to the sector* $[-k'',k'']$ *where* $k'' = \frac{k'}{\lambda'+\varepsilon}$ *for any* $\varepsilon > 0$.

The proof follows immediately from the above theorems: By Theorem 5.5 the sector $[-\frac{k'}{\lambda'}, \frac{k'}{\lambda'}]$ is the smallest sector for which the Lur'e system can possibly sustain periodic motion. Theorem 5.2 establishes that $\frac{k'}{\lambda'} \geq k^*$ is arbitrary close to the smallest upper bound k^* for which the Lur'e system is absolutely stable. The robustness margin $\varepsilon > 0$ guarantees absolute stability by Corollary 5.3.

With Corollary 5.7 we can approximate the largest sector for the Lur'e system (5.2) with symmetric sector bounds $[-k,k]$ as closely as we wish. However, the simple scaling of the sector by the largest real eigenvalue $\hat{\lambda}$ always yields symmetric sector bounds around the nominal system matrix $A = \frac{1}{2}(A_1 + A_2)$. Clearly, if we wish to keep, for example, the lower bound k_1 constant the condition in Corollary 5.7 cannot approximate the largest upper bound for k_2. However, absolute stability is guaranteed.

The conditions of the Corollaries 5.6 and 5.7 yield non-conservative stability results. Since the conditions are not based on the existence of any particular type of Lyapunov function we can expect that we can establish stability in many cases where the conditions of the Circle Criterion are violated.

However, there are some difficulties to apply the Corollaries 5.6 and 5.7 in practice.

Firstly, the conditions involve determining the greatest real eigenvalue of the infinite dimensional matrix $\boldsymbol{F}_{\omega_0}(G, \phi_r)$. This problem is addressed in the next section, where we discuss the impact of truncating the matrix $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ on the stability results. Secondly, when considering systems of order higher than three, we need to determine the greatest real eigenvalue for all periodic $\phi_r(\cdot)$ belonging to the sector $[-k, k]$ and all ratios $r > 0$. Clearly this is not possible in practice. However, the results in Section 5.3.2 encourage to search for classes of feedback modulations that are necessary and sufficient for stability.

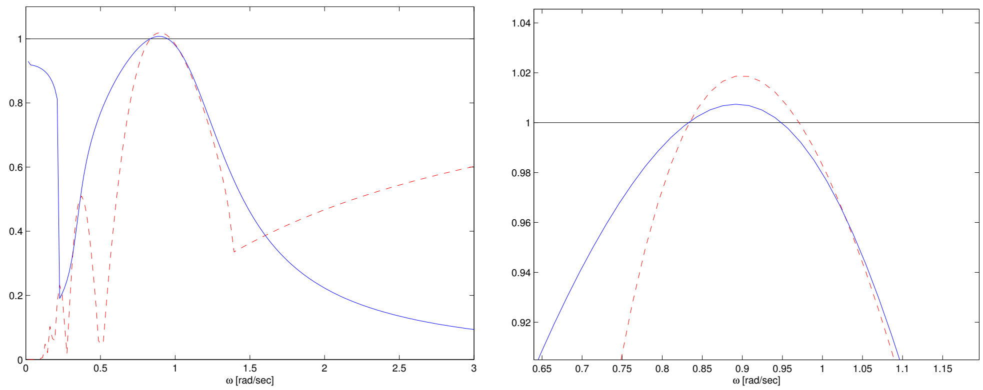
5.5 N-th order approximation

In this section we approximate the infinite dimensional matrix $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ by only considering N harmonics of the Fourier expansions of $y(t)$ and $\phi_r(t)$ and discuss its impact on the greatest real eigenvalue. Ignoring higher harmonics simply results in truncating the matrix $\boldsymbol{F}_{\omega_0}(G, \phi_r)$. Thus we only consider finite inner matrices $\boldsymbol{F}^N_{\omega_0}(G, \phi_r) \in \mathbb{C}^{(rN+1)\times(rN+1)}$.

Consider the truncated matrix $\boldsymbol{F}^N_{\omega_0}(G, \phi_r)$ where only N harmonics are considered. For $N = 2$ and frequency ratio $r = 2$ the truncated matrix takes the form

$$\boldsymbol{F}^N_{\omega_0}(G, \phi_r) = \begin{pmatrix} G(-j2\omega_0)\nu_0 & 0 & G(-j2\omega_0)\nu_{-1} & 0 & G(-j2\omega_0)\nu_{-2} \\ 0 & G(-j\omega_0)\nu_0 & 0 & G(-j\omega_0)\nu_{-1} & 0 \\ G(0)\nu_1 & 0 & G(0)\nu_0 & 0 & G(0)\nu_{-1} \\ 0 & G(j\omega_0)\nu_1 & 0 & G(j\omega_0)\nu_0 & 0 \\ G(j2\omega_0)\nu_2 & 0 & G(j2\omega_0)\nu_1 & 0 & G(j2\omega_0)\nu_0 \end{pmatrix}.$$

Truncating the matrix $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ is equivalent to replacing the truncated entries by zeros and results in a perturbation of the spectrum. The magnitude of the Fourier coefficients of the piecewise constant function $\phi_r(t)$ tends to zero when $k\omega_0 \to \infty$. Similarly, we may assume that the transfer function $G(s)$ has low-pass characteristics such that $\left|G(j\omega)\right| \to 0$ for $\omega \to \infty$. Thus, the magnitude of the entries of $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ approaches zero the further they are located from the centre-entry $G(0)\nu_0$. Hence, the perturbation of the spectrum of $\boldsymbol{F}_{\omega_0}(G, \phi_r)$ is smaller the more harmonics are included. The following examples demonstrate that the spectrum of the truncated matrix can yield good approximations for the existence of periodic motion even for small values of N.



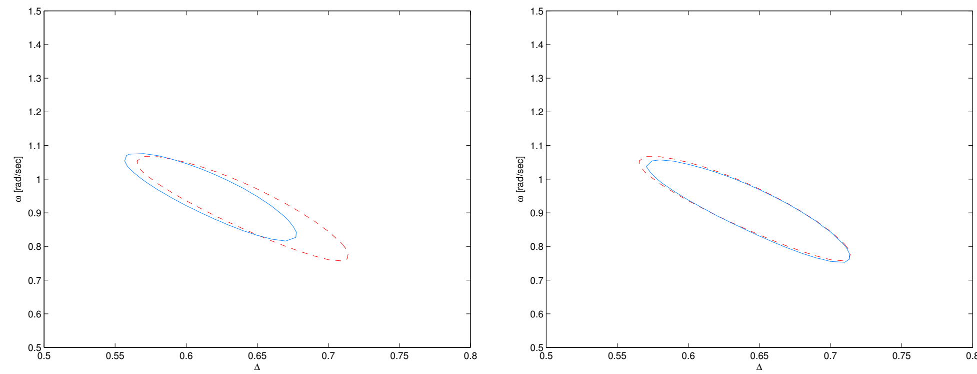
(a) (b)

Figure 5.13: Spectral radius of the transition matrix $\Phi(T,0)$ (dashed line) for feedback signals (5.6) with duty-cycle $\Delta = 0.65$ and third-order approximation (solid line) of the maximum real eigenvalue of the operator F for Example 5.3. The detail in part (b) shows that the unstable frequencies for which $\varrho\big(\Phi(T,0)\big) > 1$ are well matched by the third-order approximation.

Consider the Lur'e system in Example 5.3. To examine the quality of the approximation we shall only consider periodic functions $\phi(t)$ with two discontinuities as defined in (5.6). Recall that the first-order approximation yields the false result that there exists no periodic solution for any periodic $\phi(t)$ of the considered class (c.f. Figure 5.8). Using only three harmonics for the approximation of $\boldsymbol{F}_{\omega_0}(G,\phi_r)$ rectifies this result. Figure 5.13 shows the spectral radius $\varrho\big(\Phi(T,0)\big)$ for $\Delta = 0.65$ over the frequencies $\omega = \frac{2\pi}{T}$ (dotted line). The solid graph shows the greatest real eigenvalue of $\boldsymbol{F}^N_{\omega_0}(G,\phi_r)$ for $N = 3$ and $r = 2$ (solid line) for the same duty-cycle. Note, that the greatest real eigenvalue of $\boldsymbol{F}^N_{\omega_0}(G,\phi_r)$ does not approximate the spectral radius of the transition matrix. However, both quantities indicate the existence of periodic motion for the respective frequency of the feedback modulation $\phi(\cdot)$ when reaching a value equal to one. The detail in Figure 5.13b shows that the approximation predicts unstable solutions for switching frequencies around $0.9\frac{\text{rad}}{\text{sec}}$. In fact, the approximation of the unstable frequency-band is quite good as the comparison with the spectral radius shows. Including more harmonics increases the accuracy for this example.

To demonstrate the quality of the approximation we determine periodic solutions for the parameter space (Δ, ω_0) of the class of feedback modulations (5.6). Figure 5.14a shows the contour of duty cycles Δ and frequencies ω for which the spectral radius $\varrho\,(\Phi(T,0))$ equals one (dashed line). The solid line shows the contour for which the truncated matrix $\boldsymbol{F}_{\omega_0}(G,\phi_r)$, $r = 2$ has an eigenvalue of one. Even though the area is well matched, there is some discrepancy for duty-cycles around $\Delta = 0.7$. Using $N = 5$

harmonics for the approximation (Figure 5.14b) shows that the error is significantly reduced and the region of instability is well matched. We can observe a further increase of precision when using more harmonics for the approximation.



(a) (b)

Figure 5.14: Instability area for the system in Example 5.3 for feedback signals (5.6) over the parameter-space (Δ, ω_0). Part (a) shows the third-order approximation ; Part (b) shows the fifth-order approximation. (dashed line: spectral radius; solid line: approximation.)

Example 5.4 *Consider the switched linear system* (5.1) *with*

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.3081 & -1.4529 & -0.3097 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.4299 & -0.7557 & -1.9428 \end{pmatrix}$$

The corresponding Lur'e system is given by the transfer function

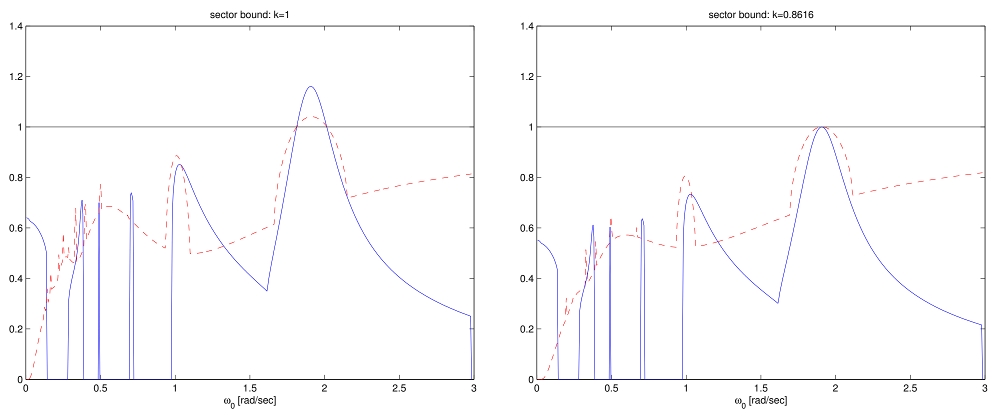
$$G(s) \quad = \quad \frac{-0.8166s^2 + 0.3486s - 0.5609}{s^3 + 1.126s^2 + 1.104s + 0.869}$$

and nonlinearity $\phi(\cdot,\cdot)$ *belonging to the sector* $[-1, 1]$.

We shall now determine the largest sector $[-k, k]$ *for which the system is absolutely stable. The transfer function* $G(j\omega)$ *has a maximum gain of* $\max_{\omega\in\mathbb{R}} |G(j\omega)| \approx 1.6$. *Hence, by the Circle Criterion (Theorem 4.19) there exists a common quadratic Lyapunov function if and only if* $k < \frac{1}{1.6} = 0.625$.

Since the system is of third order, the class of switching signals with two discontinuities is the "worst" in terms of proximity to the stability boundary [Bar93, Rap94]. Thus it is sufficient to approximate the operator for functions $\phi(\cdot)$ *in that class. For the approximation of the greatest real eigenvalue we shall again assume the frequency ratio* $r = 2$. *The fifth-order approximation of the greatest real eigenvalue of* $\boldsymbol{F}_{\omega_0}(G, \phi_r)$

attains its maximum for duty-cycle $\Delta = 0.42$ shown as solid line in Figure 5.15a. For comparison the spectral radius of the transition matrix $\Phi(T, 0)$ is shown as a dashed line. Again we can observe that the two frequencies of $\phi(t)$ (approximately $\omega_0 = 1.8$ and $\omega_0 = 2$) for which a periodic solution exists are well approximated. The greatest real eigenvalue of $\boldsymbol{F}^N_{\omega_0}(G, \phi_r)$ for all Δ, ω is approximately $\hat{\lambda} = 1.1606$. Thus, by Theorem 5.5 the smallest sector for which there exists a periodic solution is approximately $k'' = \frac{1}{\hat{\lambda}} = 0.8616$. Neglecting the approximation error, the Lur'e system is absolutely stable for any sector bound less than k''. Note, that this sector is more than a third larger than that obtained using the Circle Criterion.



(a) sector bound: $k = 1$. (b) sector bound: $k = 0.8616$

Figure 5.15: Spectral radius of the transition matrix $\Phi(T, 0)$ (dashed line) and greatest real eigenvalue of the fifth-order approximation of the operator F (solid line) for switching signals with duty-cycle $\Delta = 0.42$.

For verification, the spectral radius of the transition matrix $\Phi_\sigma(T, 0)$ is shown in Figure 5.15b for $\phi(\cdot)$ belonging to the sector $[-k'', k'']$. Again we chose the duty-cycle $\Delta = 0.42$ which yields the greatest spectral radius. By computation we obtain a greatest spectral radius of $\varrho = 1.0006$. Thus the Lur'e system is just unstable for such sector bounds. This deviation is certainly due to the truncation of the operator $\boldsymbol{F}_\omega(G, \phi_r)$. However the approximation of the smallest sector with periodic motion is very accurate given that we only used a fifth-order approximation. □

For constructing the truncated matrix $\boldsymbol{F}^N_{\omega_0}(G, \phi)$ in the above examples we only considered frequency ratios $r = 2$. This proved successful for all examples considered in the work for of this thesis. In fact, using larger frequency ratios yielded the same results. Such an approach is supported by results for the Mathieu equation for which has been found that the frequency ratio does not exceed $r = 2$ [NT73]. However, resolving this problem should be the subject of future research.

The above examples show that the open-loop operator can be successfully approximated by the truncation of higher-order terms. However, a proof of convergence of the eigenvalues of the truncated matrix to the eigenvalues of the operator has still to be found. Alternatively, it would be desirable to determine an upper bound for the approximation error on the maximum real eigenvalue induced by this truncation.

5.6 Observations on the switching frequency

For the analysis in this chapter we considered the absolute stability of the Lur'e system (5.2). However, the obtained results are also valid for the corresponding class of switched systems (5.1). In this section we briefly note some observations concerning systems that are not stable for arbitrary switching signals; in particular, we shall discuss some qualitative properties of classes of time-constrained switching signals. The best known results for this stability problem are the multiple-Lyapunov function approach and the dwell-time problem (c.f. Section 2.6). Both approaches require that the constituent systems are stable. Therefore slow enough switching will clearly result in asymptotically stable trajectories. The conditions in both approaches essentially exploit this property and commonly lead to results where switching is (at least on average) slow enough to guarantee stability. While this is certainly an interesting and practical approach, we shall see that the analysis of periodic switching signals suggest that for certain systems the converse is also true, i.e. that fast enough switching will also result in asymptotically stable trajectories.

Consider the switched linear system (5.1) with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2.0 & -1.0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -200 & -10 \end{pmatrix} \tag{5.24}$$

and the class of periodic switching signals with two discontinuities per period:

$$\sigma(t) = \begin{cases} 1 & \text{for} \quad 0 \le t < \Delta T \\ 2 & \text{for} \quad \Delta T \le t < T \end{cases}$$

Figure 5.16 shows the unstable switching signals of the above class parameterised by the duty-cycle Δ and frequency $\omega = \frac{2\pi}{T}$. The solid lines shows the level curves for which the spectral radius of the transition matrix $\Phi(t, 0)$ equals one. We can observe that there are several regions of instability. In fact, for some duty-cycles $\Delta \in [0, 1]$ there are a number of frequency bands for the switching signal that result in unstable trajectories. In Figure 5.17 the spectral radius is shown for $\Delta = 0.7$. As expected, the switched system is stable for low switching frequencies. There are two instability bands for frequencies between roughly 5 and $20\,\frac{\mathrm{rad}}{\mathrm{s}}$. For higher frequencies, however, the spectral radius converges to one. It appears that there are a number of instability bands for certain mid-range frequencies, but the system remains stable for frequencies higher than $21\,\frac{\mathrm{rad}}{\mathrm{s}}$.

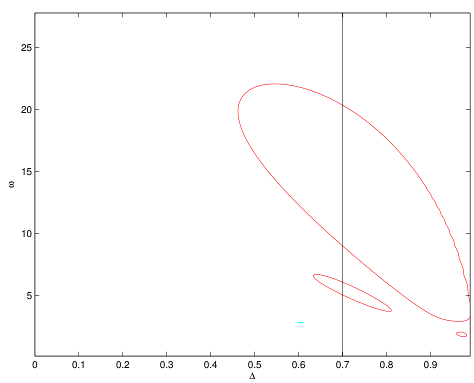


Figure 5.16: Unstable switching signals with two switches per period. The figures shows the level curves of the spectral radius equal to one.

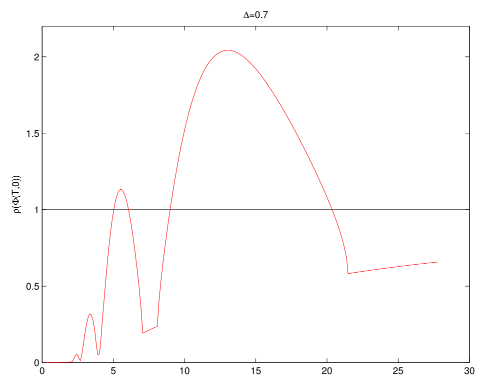


Figure 5.17: Frequency-bands of instability for a switching signal with two switches per period and duty-cycle $\Delta = 0.7$.

In fact we can observe that for any given $\Delta \in [0, 1]$ the trajectories of switched system converge to those of the LTI system Σ_A with system matrix $A = \Delta A_1 + (1 - \Delta)A_2$. Figure 5.18 shows a trajectory for the switched system for $\Delta = 0.7$ and $\omega = 90$. The trajectory of the switched system (solid line) chatters along the trajectory of the average LTI system Σ_A (dashed line). For higher frequencies the switched system attains a sliding mode-like behaviour. This observation implies that switched systems where the convex combination $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$ are stable for very low and very high switching frequencies. In between there can be a number of instability bands as shown in Figure 5.17. Conversely, it follows that a sufficient condition for the instability of the switched system is that the convex combination $\alpha A_1 + (1 - \alpha)A_2$ is non-Hurwitz for some $\alpha \in [0, 1]$.

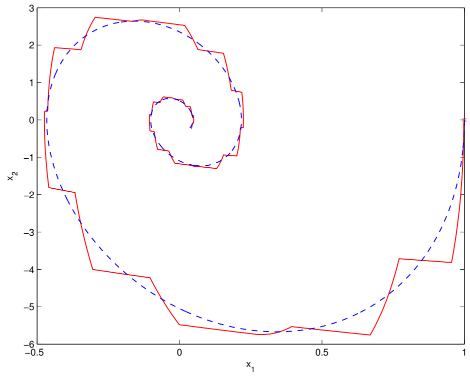


Figure 5.18: Sample trajectory for high switching frequency with duty-cycle $\Delta = 0.7$ (solid line), and trajectory of the LTI system Σ_A.

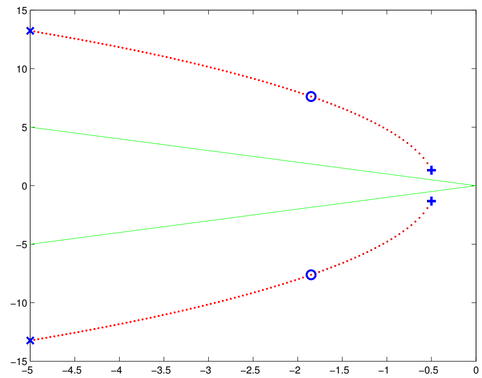


Figure 5.19: Eigenvalue-locus $\sigma_\alpha[A_1, A_2]$ for system (5.24). The eigenvalue for $\alpha = 0.7$ are indicated by the circles.

5.7 Discussion and summary of the results

In this chapter we considered the stability of the class of switched systems that can be represented in Lur'e type form. The main focus of the analysis was on finding conditions for which the system has a periodic solution. The approach in this chapter is motivated by the stability conjecture of Power&Tsoi and Shorten. The validation of this conjecture raised a number of questions on the relation of periodic signals and absolute stability. Perhaps the most important issue in this analysis is to identify a class of feedback signals that is necessary and sufficient for deriving conditions for absolute stability. Or simply speaking, to identify the "worst" feedback signal for a given sector.

Characteristic of the worst feedback modulation

The absolute stability problem can be reformulated as the problem of finding the nonlinearity $\phi^*(\cdot,\cdot)$ that is in some sense most destabilising for the system. In this chapter we characterised some properties of such a class. By equivalence of absolute stability of Lur'e systems and uniform asymptotic stability of switched linear systems, we can identify that $\phi^*(\cdot,\cdot)$ is a piecewise constant gain $\phi^* : \mathbb{R} \to \{k_1, k_2\}$. Further it has been established that periodic motion is significant in the sense that we can approximate the stability boundary k^* arbitrarily close by considering the behaviour of the system under periodic feedback signals.

For second and third order systems $\phi^*(\cdot)$ has two discontinuities per period [Pya71, Bar93, Rap94]. We have shown that this also holds for the first-order approximation for systems of order $n > 3$. This fact together with empirical studies suggests that $\phi^*(t)$ might indeed be periodic and has two discontinuities per period. If this were true, it would considerably simplify the stability analysis of Lur'e systems.

Stability conditions

Following the ideas first introduced by Power & Tsoi and later by Shorten, we derived a necessary and sufficient condition for the existence of periodic motion for a given periodic feedback signal. Using the results of the analysis of the existence of periodic motion in proximity of the stability boundary we derived sufficient conditions for absolute stability that exceed those stability bounds obtained from Circle Criterion. Moreover, we can directly identify the smallest (symmetric) sector for which a periodic solution exists. This implies in turn that we can approximate the largest sector for which the Lur'e system is absolutely stable as closely as we wish.

Unfortunately, the conditions can not be directly applied without further analysis. Firstly, the conditions involve the determination of the eigenvalues of an infinite dimensional matrix (which is clearly not possible). However, since the considered strictly proper transfer function $G(s)$ has low-pass characteristics, the infinite dimensional matrix can, at least in some circumstances, be approximated by a low dimensional matrix. Further, we demonstrate by example that such procedure can lead to satisfying results for some classes of problem. However, a proof of convergence of the approximation procedure that we introduce, as well as an error bound for the approximation, has yet to be found.

The second problem for applying the results in practice is due to the large number of matrices that need to be checked, since every switching signal and every frequency ratio of output $y(\cdot)$ and periodic feedback modulation $\phi(\cdot)$ results in a separate matrix. Clearly, resolving the problem of identifying the "worst" feedback signal would simplify this issue. There are also some indications for finding the correct frequency ratio. For the Mathieu equation it can be shown that the frequency ratio is at most $r = 2$ [NT73]. This is confirmed by observations in [PT74] as well as empirical studies for this thesis.

It should be noted that the numerical application of the results in this chapter require more computational effort than calculating the spectral radius of the transition matrix. However, the construction of the open-loop operator gives a number of analytic insights for future investigations; in particular in view of identifying the worst feedback modulation. Further, determining the transition matrix for *continuously* varying systems is by no means trivial. The open-loop operator could potentially be constructed for periodic feedback modulations that are continuous since it only depends on the frequency-spectrum of the feedback modulation signal. Thus the analysis of this chapter could also be applicable to classes of continuously time-varying systems. However, further research on this problem is needed. Last but not least, the open-loop operator approach allows us to determine the smallest sector of nonlinearities for which periodic motion is possible. Formulated as a sufficient condition for stability this yields a certain robustness measure for the absolute stability of the Lur'e system.

Chapter 6

Control-design approach for switched plants with N modes

In this chapter we consider the control-design for a process with switched dynamics of arbitrary order and N modes. The approach is motivated by the control-objective that requires the closed-loop behaviour to be similar for each mode. We propose a switched controller structure to approach this design task. Based on the structure of the resulting closed-loop system, sufficient conditions for stability are derived. Furthermore, the transient behaviour of the closed-loop system at the switching instances is analysed and design criteria for transient-free switching under certain conditions are derived. It is shown that closed-loop stability and transient-free switching can be achieved simultaneously.

6.1 Introductory remarks

Thus far we have considered stability of several classes of switched linear systems and used different approaches to obtain stability conditions for arbitrary switching. In this chapter we shall move the focus from the pure analysis of the given switched system towards a control-design procedure that ensures stability of the switched system as well as certain design-objectives. Moreover, we extend the analysis from switched systems with two modes to systems with a finite number of subsystems.

The stability analysis of switched systems with N constituent systems is much more complex than the analysis of systems with only two subsystems. Clearly, the switched system with N constituent systems is asymptotically stable if all subsystems Σ_{A_i}, $i \in \mathcal{I} = \{1, \ldots, N\}$ share a single common Lyapunov function. However it is shown in [SN00] that the existence of a CQLF for every pair of system $\left(\Sigma_{A_i}, \Sigma_{A_j}\right), \forall\, i, j \in \mathcal{I}$ is not sufficient for the existence of a single CQLF for *all* subsystems. In fact, the authors demonstrate by example that such a switched system can even be unstable for some switching signals and thus no common Lyapunov function exists.

Most of the available algebraic conditions for asymptotic stability of switched systems with N subsystems depend on the structure of the constituent matrices. For example, it is well known that the LTI systems Σ_{A_i}, $i \in \mathcal{I}$ share a CQLF if the system matrices A_i are symmetric or commute pairwise. Moreover, Theorem 2.17 establishes that the LTI systems Σ_{A_i} with $A_i \in \mathcal{A}$ have a CQLF if the matrices A_i are Hurwitz and are simultaneously similar to triangular matrices, i.e. there exists a non-singular matrix T such that $\tilde{A}_i = TA_iT^{-1}$ is in upper triangular form for all $i \in \mathcal{I}$. Unfortunately, system matrices of closed-loop control systems are in general not in such form. Therefore, this result can only be applied to a restricted class of control systems where the closed-loop system matrices are naturally in triangular form or are simultaneously similar to triangular matrices. The latter is usually not easy to show (see [Laf78] for conditions for simultaneously similar triangular matrices). In this chapter we shall consider system matrices whose structure arises naturally from a general switched controller architecture. The structure of the resulting closed-loop system matrices is then exploited to derive sufficient conditions for stability.

Besides the stability problem, switched linear systems exhibit also additional transient behaviour unknown to linear time-invariant systems. While LTI systems only have transient responses when any of the input signals (including disturbances) changes, switched systems can also exhibit transient behaviour at the switching instances. To illustrate such effect consider the switched system $\Sigma_{\mathcal{A}}$ with $\mathcal{A} = \left\{A_1, A_2\right\}$ where

$$A_1 = \begin{pmatrix} -1 & 7 \\ -1 & -9 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -9 & 7 \\ -1 & -1 \end{pmatrix}. \tag{6.1}$$

As shown in Figure 6.1, the eigenvalue-locus of the matrix-pencil $\sigma_\alpha[A_1, A_2]$ lies completely within the 45°-Region. (Note that the eigenvalues of A_1 and A_2 are identically given by $\{-2 - 8\}$.) Hence, by Corollary 3.19 the switched system $\Sigma_{\mathcal{A}}$ is asymptotically stable for arbitrary switching.

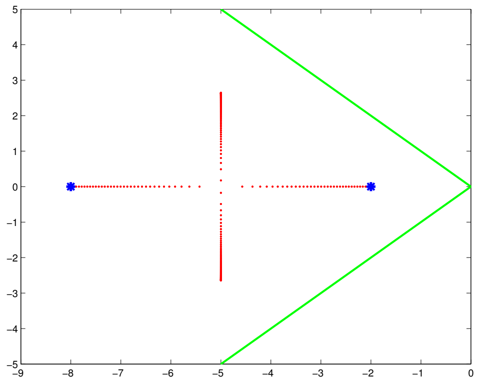


Figure 6.1: Eigenvalue-locus of the switched system $\Sigma_{\mathcal{A}}$ with system matrices (6.1).

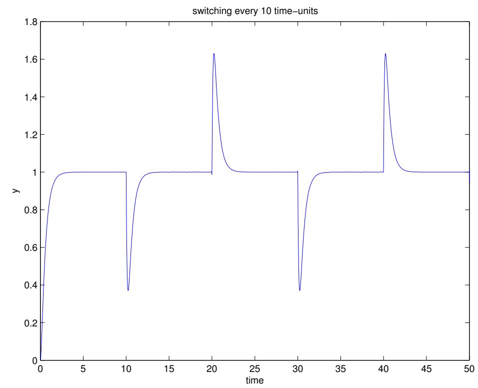


Figure 6.2: Step-response for switching at every 10 time-units

Consider now the switched input-output system (2.4) with system matrices (6.1), input-vectors $b_1 = b_2 = \begin{pmatrix} 0 & 2.2857 \end{pmatrix}^{\mathsf{T}}$, and output-vectors $c_1^{\mathsf{T}} = c_2^{\mathsf{T}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. This input-output system is BIBO stable, since the autonomous switched system is asymptotically stable [Rug96].

The dynamics of the SISO system are identical for each mode and given by the transfer function

$$G(s) \quad = \quad \frac{16}{s^2 + 10s + 16}\,.$$

Figure 6.2 shows the output response for a unit-step input of the switched input-output system where switching occurs every 10 time-units. We observe, that the switched system exhibits large transient responses whenever the system mode changes. Clearly, such behaviour is not desirable for a closed-loop control system.

Beside the stability of the closed-loop system, its transient behaviour is perhaps the most important design criterion for the controllers for processes with switched dynamics. In this chapter we consider a class of switched input-output systems and propose a controller structure to achieve stability of the closed-loop switched system as well as convenient transient behaviour. In the remainder for this section the control problem considered is specified and the switched controller that we shall apply is introduced. In the following section we analyse the stability of the resulting closed-loop system. In Section 6.3 we derive conditions for which the closed-loop system has no transient responses when switching in steady state. In the last section we relate the switched controller considered in this chapter to alternative approaches.

6.1.1 Problem description

The approach chosen in this chapter is motivated by a typical control-design task for switched systems. For a given process with switching linear dynamics, a controller has to be found such that the closed-loop system is stable and its dynamical behaviour is similar in every plant-mode. Thus the mode-switches of the plant are interpreted rather as a disturbance to the process than a change of control objectives.

We consider time-varying SISO plants with piecewise constant dynamics and N modes. At any time-instant the dynamics are described by one of the non-autonomous LTI systems $\big(A_i, B_i, C_i\big)$, $i \in \mathcal{I} = \big\{1, \dots, N\big\}$. Thus the plant dynamics are given by the non-autonomous switched system

$$\dot{x}_p(t) = A_{\sigma(t)} x_p(t) + B_{\sigma(t)} u(t) \tag{6.2a}$$

$$y(t) = C_{\sigma(t)} x_p(t) \tag{6.2b}$$

with arbitrary switching signals $\sigma : \mathbb{R}^+ \to \mathcal{I}$. The plant-state is denoted $x_p \in \mathbb{R}^{n_p}$, the input $u(t)$ and output $y(t)$ are scalar. We require that the LTI systems $\big(A_i, B_i, C_i\big)$ of each mode $i \in \mathcal{I}$ are completely controllable and are in the control canonical form

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -q_{0i} & -q_{1i} & \cdots & \cdots & -q_{n_p-1,i} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{6.3}$$

and $C_i = \big(p_{10} \;\; \cdots \;\; p_{n_p-1,i}\big)$.

With each mode $i \in \mathcal{I}$ we associate the transfer function

$$P_i(s) = \frac{p_{n_p-1,i} s^{n_p-1} + \ldots + p_{1i} s + p_{0i}}{s^{n_p} + q_{n_p-1,i} s^{n_p-1} + \ldots + q_{1i} s + q_{0i}}$$

that describes the dynamics of the LTI system $\big(A_i, B_i, C_i\big)$.[1] We shall assume that the mode-switches of the plant are immediately detectable such that the switching instances can be assumed to be known for the controller.

[1] Note that the transfer functions are only instantaneous descriptions of the plant dynamics. They do not represent the time-varying differential equation (6.2), but are only valid for time-instances where the dynamics of the plant are constant. Therefore we may only use the transfer function description for the control-design, but not for the stability analysis.

The objective is to design a controller such that the closed-loop system

- is asymptotically stable for arbitrary switching signals,
- has the poles $\Lambda_t \subset \mathbb{C}_-$, specified independently of the plant mode $i \in \mathcal{I}$,
- has minimal transient responses at the switching instances.

Before we introduce the switched controller considered in this chapter a remark on the realisation of the transfer functions is order. When discussing properties of the transfer functions $P_i(s)$ of the constituent systems we shall bear in mind that we cannot choose their realisations $\big(A_i, B_i, C_i\big)$ independently of each other. The realisations are rather given by (6.3) or by *simultaneous* co-ordinate transformations of all modes $i \in \mathcal{I}$.

6.1.2 Control approach

In order to achieve the design objectives we associate an individual controller for each plant mode $i \in \mathcal{I}$. We choose a controller architecture where each controller is realised as a time-invariant LTI system as depicted in Figure 6.3. We refer to this architecture of the switched controller as *local-state controller*.

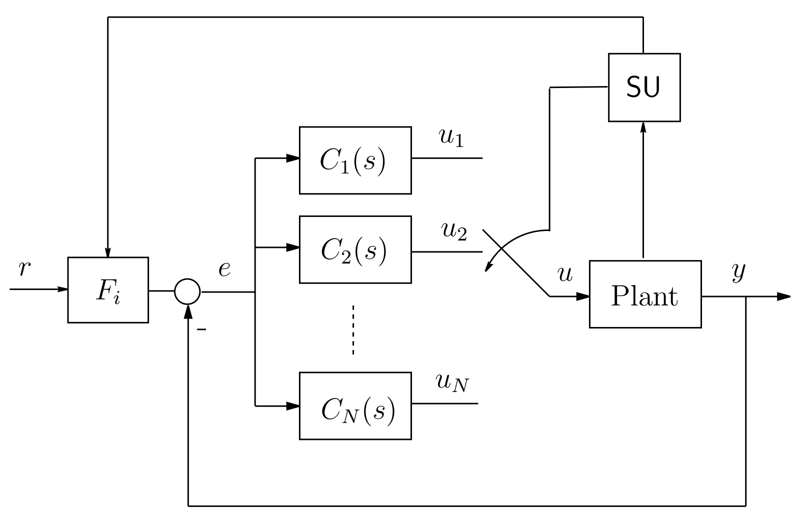


Figure 6.3: Local-state controller architecture.

At the switching instances of the plant the switching unit (SU) connects the plant-input u with the respective controller-output u_i of the associated controller and adjusts the respective pre-filter gain F_i (when applicable). Thus, at any given point in time, only one of the N controllers is active in the closed loop. We shall assume that there is no time-delay between the switching of the plant and the switching action of the control-signal such that whenever the plant is in mode i the respective controller $C_i(s)$ is active in the loop.

The dynamics of the individual controllers of the local-state controller are given by

$$\begin{aligned}\dot{x}_i(t) &= K_i x_i(t) + L_i e(t) \\ u_i(t) &= M_i x_i(t) + J_i e(t)\end{aligned}$$

where $x_i(t) \in \mathbb{R}^{n_c}$ is the state-vector of the controller associated with mode $i \in \mathcal{I}$; the input $e(t) \in \mathbb{R}$ is shared by all controllers $i \in \mathcal{I}$ and each controller has an individual control signal $u_i(t) \in \mathbb{R}$. The dimensions of the controller realisations are $K_i \in \mathbb{R}^{n_c \times n_c}$, $L_i, M_i^\mathsf{T} \in \mathbb{R}^{n_c}$ and $J_i \in \mathbb{R}$.

An important property of this controller architecture is that each of the individual controllers has their own state-vector x_i. Thus, the state-vector of the switched closed-loop system consists of the plant states x_p and the local states x_i of the individual controllers, i.e. $x = \begin{pmatrix} x_p^\mathsf{T} & x_1^\mathsf{T} & \cdots & x_N^\mathsf{T} \end{pmatrix}^\mathsf{T}$. The autonomous closed-loop system is then given by

$$\dot{x}(t) = H(t)x(t), \qquad H(t) \in \mathcal{H} = \{H_1, \dots, H_N\} \subset \mathbb{R}^{n \times n} \tag{6.4a}$$

where $n = n_p + N n_c$. The local-state controller yields the closed-loop system matrices

$$H_i = \begin{pmatrix} A_i - B_i J_i C_i & B_1 M_1 \delta_{i1} & \cdots & \cdots & B_N M_N \delta_{iN} \\ -L_1 C_1 & K_1 & 0 & \cdots & 0 \\ -L_2 C_2 & 0 & K_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -L_N C_N & 0 & \cdots & 0 & K_N \end{pmatrix} \tag{6.4b}$$

for all $i \in \mathcal{I}$, where δ_{ij} is the Kronecker symbol.

Note that the last $N n_c$ rows of H_i are equal for all $i \in \mathcal{I}$. In fact, since the system matrices A_i of the plant are in companion form and the vector-products $B_i M_i$ have only non-zeros entries in n_p-th row, all matrices $H_i \in \mathcal{H}$ are identical except for the n_p-th row.

For the controller design we use the standard pole-placement technique for LTI systems. The controller for each mode i can be represented by the transfer function

$C_i(s) = M_i(sI - K_i)^{-1}L_i + J_i$. For every mode i we design the controller $C_i(s)$ such that the poles of the active closed-loop transfer functions

$$T_i(s) \quad = \quad \frac{C_i(s)G_i(s)}{1 + C_i(s)G_i(s)} \tag{6.5}$$

are given by the set $\Lambda_t = \{\lambda_1, \ldots, \lambda_{n_p+n_c}\}$.

We shall assume that the pole placement requirement is feasible for every mode and that the target poles in Λ_t are distinct:

Assumption 6.1 *The set of target-poles Λ_t is simple, equal for all modes $i \in \mathcal{I}$ and are obtained for the closed-loop transfer function $T_i(s)$, $\forall\, i \in \mathcal{I}$, by controllers $C_i(s)$ with distinct poles in the open left half-plane.*

Thus the poles of the active closed-loop transfer function are independent of the current mode. The main objective in this chapter is to analyse the stability of the resulting closed-loop switched system and to investigate whether transient-free switching is possible.

Note that for each matrix H_i all but one of the Kronecker symbols are equal to zero. Thus the spectrum of H_i contains the eigenvalues of all non-active controllers, i.e. $\sigma\big(H_i\big) \supset \sigma\big(K_l\big)$ for $l \neq i$. By Assumption 6.1 the remaining $n_p + n_c$ eigenvalues of the matrices H_i are given by the target-poles Λ_t. Therefore the spectrum of the system matrix H_i is given by

$$\sigma\big(H_i\big) \quad = \quad \Lambda_t \cup \bigcup_{l \neq i} \sigma\big(K_l\big)\,, \qquad i \in \mathcal{I}.$$

Before we proceed, it shall be noted that there are many other possibilities to realise a switched feedback system. For instance, instead of using time-invariant controllers and switching the output-signal we could also realise the controllers as a single switched system on its own. These two control structures lead to different closed-loop systems that can have significantly different behaviour as pointed out in [LSL+03].

6.2 Stability analysis

In this section we analyse the stability of the closed loop system. For first order controllers we employ a result in [SOC01] to show that the closed loop system is always stable when the pole-placement requirement is feasible. This does not generalise for controllers of higher order. However, we can show that the stability analysis can be significantly simplified for these cases.

6.2.1 Stability using first-order controllers

In [SOC01] global attractivity of a class of switched systems with real eigenvalues is proven. In this section we shall analyse to what extent this theorem can be applied to the system class considered in this chapter.

The result in [SOC01] applies to switched systems with system matrices in $A_i \in \mathcal{A} \subset \mathbb{R}^{n\times n}$ which are constructed as follows. Given the set of $n+1$ real vectors $\mathcal{V} \subset \mathbb{R}^n$ where any subset of n vectors are linearly independent, define the non-singular matrix $V_0 \in \mathbb{R}^{n\times n}$ with columns $v_1, \ldots, v_n \in \mathcal{V}$. The columns of V_0 are linearly independent vectors and the only element in $\mathcal{V}$ not involved in constructing V_0 is v_{n+1}. We define n further matrices V_i by replacing the i-th column of V_0 by the left-out element v_{n+1}. Hence, the matrices V_i for $i = 0, \ldots, n$ are non-singular and have pairwise $n-1$ columns in common. Let $\mathcal{D}$ be a set of diagonal matrices $D_j \in \mathbb{R}^{n\times n}$ with negative, real entries on the diagonal. Choosing matrices D_j and V_i we construct the system matrices $A_{ij} = V_i D_j V_i^{-1}$. The following theorem is proven in [SOC01]:

Theorem 6.2 *Given the sets $\mathcal{V}, \mathcal{D}$ and non-singular matrices $V_i \in \mathbb{R}^{n\times n}$ constructed as above. The switched system*

$$\dot{x}(t) = A(t)x(t)$$

with $A(t) \in \mathcal{A} = \left\{A_{ij} \middle| A_{ij} = V_i D_j V_i^{-1},\ i = 1, \ldots, n+1,\ D_j \in \mathcal{D}\right\}$ is asymptotically stable for arbitrary switching signals.

The proof of the theorem has some very interesting features. It does not directly depend on the existence of any type of common Lyapunov function for the systems $\Sigma_{A_{ij}}$. The authors rather embed the system matrices A_{ij} in an augmented state-space such that one eigenvector is common to *all* augmented matrices $\tilde{A}_{ij}$. By projecting the state-trajectories onto that common eigenvector it can be established that the switched system is asymptotically stable.

In this section we shall show that Theorem 6.2 can be applied to establish asymptotic stability of a class of switched systems of the form (6.4). The conditions of Theorem 6.2 require the following:

- the system matrices H_i are individually diagonalisable;
- the eigenvalues of the system matrices H_i are negative real for all $i \in \mathcal{I}$;
- every pair of system matrices (H_i, H_j) has $n-1$ common eigenvectors.

Before we proceed applying Theorem 6.2 to the switched system (6.4) we shall note a useful implication of the pole-placement approach (Assumption 6.1). A consequence of this approach is that the subspace H_i corresponding to the target poles Λ_t does not depend on i. This is the content of the following lemma.

Lemma 6.3 *Let $\lambda \in \Lambda_t$ be a simple eigenvalue of each $H_i, i \in \mathcal{I}$, then there exists a vector $v \neq 0$ such that*

$$H_i v \quad = \quad \lambda v \qquad for\ all\ i \in \mathcal{I}. \tag{6.6}$$

Proof. The eigenvector of H_i corresponding to the eigenvalue λ lies in the null-space of $\tilde{H}_i = \lambda I - H_i$. By inspection we find that the first $n_p - 1$ rows and the last Nn_c rows of $\tilde{H}_i$ are linearly independent. Thus the eigenvector $v_i \in \mathbb{R}^n$ satisfying $\big(\lambda I - H_i\big)v_i = 0$ is orthogonal to these $n-1$ rows and therefore completely specified in $\mathbb{R}^n$. Consider now the null-space of $\tilde{H}_j = \lambda I - H_j$ for the same eigenvalue $\lambda \in \Lambda_t$. The first $n_p - 1$ rows and the last Nn_c rows of $\tilde{H}_i$ and $\tilde{H}_j$ are identical. Therefore the eigenvector v_j of H_j corresponding to λ is orthogonal to exactly the same $n-1$ linear independent vectors as v_i. Thus v_i and v_j are co-linear and the matrices $H_i \in \mathcal{H}$ have the eigenvector v in common. $\square$

The above lemma implies that the matrices $H_i \in \mathcal{H}$ have $n_p + n_c$ common eigenvectors that correspond to the target poles Λ_t.

Using the same arguments as in the above proof it can be shown that every pair of matrices $\big(H_i, H_j\big)$ has also $(N-2)n_c$ eigenvectors in common that correspond to the common non-active controllers $K_l,\ l \neq i, j$. Thus all matrices in $\mathcal{H}$ have $n_p + n_c$ common eigenvectors and every pair of matrices in $\mathcal{H}$ has $n - n_c$ common eigenvectors.

Let the eigenvalues in Λ_t be distinct and real and choose controllers of first order. Under the assumption that the pole-placement problem is feasible and results in stable controllers (Assumption 6.1), the requirements of Theorem 6.2 are satisfied.

Corollary 6.4 *Consider the switched linear plant* (6.2) *with N modes of order n_p and let $\Lambda_t \subset \mathbb{R}^-$ be the set of distinct target-poles. If there exist N stable first-*

order controllers that solve the pole-placement problem (6.5) *for every mode, then the resulting switched system* (6.4) *is asymptotically stable for arbitrary switching.*

The corollary follows by applying Lemma 6.3 and Theorem 6.2. Since the eigenvalues of H_i are distinct and real, the matrices can be decomposed as $H_i = V_i D_i V_i^{-1}$. Every pair of matrices (H_i, H_j) has $n_p + 1$ eigenvectors in common that correspond to the target-poles Λ_t and $N - 2$ eigenvectors corresponding to the poles of the common non-active controllers in common. The matrices D_j in Theorem 6.2 are only required to have real negative entries on the diagonal. Hence, we can individually arrange the eigenvectors for all i such that the matrices V_i satisfy the structural requirements of the Theorem. Defining the sets $\mathcal{V} = \{V_i,\ i \in \mathcal{I}\}$ and $\mathcal{D} = \{D_i,\ i \in \mathcal{I}\}$ we can apply Theorem 6.2.

Corollary 6.4 guarantees asymptotic stability for a class of switched systems of the form (6.4) with an arbitrary number of subsystems. However, it is restricted by the requirement that the controllers are of first order and the pole-placement problem is feasible for stable controllers. Under normal circumstances, this restricts the class of plants to order $n_p = 2$ or less [GGS01]. In the next section we shall consider switched plants of higher order and derive conditions that guarantee stability of the switched closed-loop system.

6.2.2 Stability for controllers of arbitrary order

In this section we extend our analysis to plants of arbitrary order $n_p > 2$. In order to meet the pole-placement requirement we will need to apply controllers of order up to $n_c = n_p - 1$, [GGS01]. Thus the closed-loop system (6.4) is at most of order $n = n_p + N(n_p - 1)$.

We approach the stability analysis by considering the common subspaces of the constituent system matrices in $\mathcal{H}$. Recall that all matrices in $\mathcal{H}$ have $n_p + n_c$ eigenvectors in common and every pair (H_i, H_j) has $n - n_c$ eigenvectors in common. We begin our analysis by studying the case where only two subsystems are present ($N = 2$).

Two constituent systems

In this section we consider asymptotic stability of the switched system (6.4) with $i \in \mathcal{I} = \{1, 2\}$ and plant dynamics Σ_{A_i} of order $n_p > 2$. The closed-loop system matrices H_i are given by

$$H_1 = \begin{pmatrix} A_1 - B_1 J_1 C_1 & B_1 M_1 & 0 \\ -L_1 C_1 & K_1 & 0 \\ -L_2 C_2 & 0 & K_2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} A_2 - B_2 J_2 C_2 & 0 & B_2 M_2 \\ -L_1 C_1 & K_1 & 0 \\ -L_2 C_2 & 0 & K_2 \end{pmatrix}. \quad (6.7)$$

Due to the pole-placement requirement we get for the respective spectra

$$\begin{aligned} \sigma\big(H_1\big) &= \Lambda_t \cup \sigma\big(K_2\big) \\ \sigma\big(H_2\big) &= \Lambda_t \cup \sigma\big(K_1\big). \end{aligned}$$

Let the columns of $V_t \in \mathbb{C}^{n\times(n_p+n_c)}$ form a basis of the common subspace of H_1 and H_2, and consider the matrix

$$T_1 = \begin{pmatrix} V_t & e_{(n_p+n_c+1)} & \dots & e_n \end{pmatrix}$$

where e_k denotes the k'th unit vector in $\mathbb{R}^n$. Note that T_1 is invertible as the vectors $e_{(n_p+n_c+1)}, \dots, e_n$ form a basis of an invariant subspace of H_1, which does not intersect $\operatorname{span} V_t$ as $\Lambda_t \cap \sigma\big(K_2\big) = \emptyset$.

Applying the similarity transformation T_1 to our two system matrices we obtain

$$T_1^{-1} H_1 T_1 = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} \quad (6.8a)$$

$$T_1^{-1} H_2 T_1 = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} + \begin{pmatrix} 0_{(n_p+n_c)\times(n_p+n_c)} & U_1 \\ 0_{n_c\times(n_p+n_c)} & U_2 \end{pmatrix} \quad (6.8b)$$

where $\sigma\big(D_t\big) = \Lambda_t$. Note that $rank\{U_2\} = 1$ as we have $rank\big\{H_1 - H_2\big\} = 1$. Further it follows from the spectrum of H_2 that $\sigma\big(K_2 + U_2\big) = \sigma\big(K_1\big)$.

The following theorem reduces the stability problem of the switched system defined by $\{H_1, H_2\}$ to a stability problem only involving the controllers.

Theorem 6.5 *Consider the matrices H_1, H_2 in (6.7) and let Assumption 6.1 be satisfied such that $\sigma\big(H_i\big) = \Lambda_t \cup \sigma\big(K_j\big)$ for $i, j = 1, 2,\ i \neq j$. Assume furthermore that $\Lambda_t \cap \sigma\big(K_i\big) = \emptyset$, $i = 1, 2$. Then the following statements are equivalent:*

(i) the switched system given by the set of matrices $\{H_1, H_2\}$ is asymptotically stable for arbitrary switching signals;

(ii) the switched system given by the set of matrices $\{K_2, K_2+U_2\}$ *is asymptotically stable for arbitrary switching signals;*

(iii) the switched system given by the set of matrices $\{K_1, K_2\}$ *is asymptotically stable for arbitrary switching signals.*

Proof. The equivalence of (i) and (ii) can be seen as follows. Firstly, the matrices in (6.7) and (6.8) are obtained from one another by similarity. Thus the set $\{H_1, H_2\}$ defines an asymptotically stable switched system if and only if $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ does. On the other hand $\Lambda_t \subset \mathbb{C}_-$, so that the exponential stability of $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ is equivalent to that of the lower diagonal block $\{K_2, K_2+U_2\}$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows if we find a similarity transformation that transforms K_2 and K_2+U_2 into K_2 and K_1 respectively. Note first, that since *rank*$\{H_2 - H_1\} = 1$, the perturbation $(U_1^\intercal, U_2^\intercal)^\intercal$ is also of rank one. Further, the block K_2+U_2 is similar to K_1 since the eigenvalues in Λ_t are generated by the closed loop system of A_2 and K_2.

Consider now the matrices $K_2^\intercal$ and $K_2^\intercal + U_2^\intercal$. If we can find a vector x such that

$$x_m := (K_2^\intercal)^m x = (K_2^\intercal + U_2^\intercal)^m x\,, \text{ for } m = 0, \ldots, n-1\,, \tag{6.9}$$

and so that the sequence $x_m, m = 0, \ldots, n-1$ is linearly independent, then the similarity transformation

$$T = \begin{pmatrix} x_0 & \ldots & x_{n-1} \end{pmatrix}$$

yields

$$T^{-1}K_2^\intercal T = K_2^\intercal \quad \text{and} \quad T^{-1}(K_2^\intercal + U_2^\intercal)T = K_1^\intercal\,,$$

as the assumption (6.9) guarantees that both matrices are brought simultaneously in transposed companion form (sometimes also known as second companion form) and because the companion form of K_2+U_2 is K_1 by similarity. By taking transposes of the previous equations we have found the desired transformation that concludes the proof in case that (6.9) holds. Now by induction the conditions in (6.9) require that

$$U_2^\intercal (K_2^\intercal)^m x = 0 \;\text{ for }\; m = 0, \ldots, n-2\,.$$

As *rank*$\{U_2^\intercal\} = 1$ the kernel of $U_2^\intercal (K_2^\intercal)^m$ has dimension $n-1$ for $m = 0, \ldots, n-2$ and so by dimensionality reasons the intersection of these kernels satisfies

$$V \;:=\; \bigcap_{m=0}^{n-2} \ker U_2^\intercal (K_2^\intercal)^m\,, \quad \dim V \;\geq\; 1\,.$$

Choose an $x \in V$, $x \neq 0$. If the set of vectors $\{x_m, m = 0, \dots, n-1\}$ is linearly independent, then (6.9) holds and we are done. If this is not the case this means that the lower-dimensional K_2^{T}-invariant subspace defined by

$$W := \operatorname{span}\{x_m \mid m = 0, \dots, n-1\}$$

is by definition contained in the kernel of U_2^{T}. Hence on this lower dimensional subspace K_2^{T} is not perturbed by U_2^{T}. We may then repeat the argument on the restriction of K_2^{T} to a complementary invariant subspace and repeat the argument until (6.9) holds on one of this lower dimensional complementary subspaces. The procedure terminates for reasons of dimensionality and the assertion follows. □

In the previous section it was shown that our suggested design-procedure, when feasible for first-order controllers, always results in asymptotically stable systems for arbitrary switching. Unfortunately, this does not hold for controllers of higher order. However, Theorem 6.5 provides a convenient simplification of the stability analysis. To guarantee asymptotic stability of the switched system (6.4) with $N = 2$ we only need to consider the asymptotic stability for arbitrary switching signals of the autonomous switched system given by the controller matrices:

$$\dot{x}(t) = K(t)x(t), \qquad K(t) \in \{K_1, \dots, K_N\} \subset \mathbb{R}^{n_c \times n_c}. \tag{6.10}$$

Thus, the stability problem of the switched system (6.4) of order $n_p + 2n_c$ is reduced to the stability problem of a switched system of order n_c.

It should be emphasised that the proof of Theorem 6.5 relies on the fact that the controller-matrices are in companion form. At this point it is not clear what role the specific realisation chosen for the controllers plays for the result. However, it is obvious that the equivalence $(ii) \Leftrightarrow (iii)$ can only be true when $rank\{K_1 - K_2\} = rank\{U_2\} = 1$.

The equivalence of the asymptotic stability of the system (6.10) and (6.4) is in actual fact less obvious than intuition might suggest. As we shall see in the next section, the result does not generalise for systems with more than two subsystems. In this context it might be worth noting that the switched system (6.10) is actually not part of the closed-loop system (6.4) – at least not explicitly. For the switched system (6.10) the controller dynamics K_i act on the same state-space, however the controllers in the closed-loop system (6.4) are realised as local-state controllers and therefore do not share the states.

Stability of N constituent systems

We shall now discuss the reduction in complexity that can be obtained for switched systems with $N > 2$ modes. Unfortunately, the proof of Theorem 6.5 uses in a very specific way the fact that only two subsystems are involved. The following example shows that the asymptotic stability of the switched system (6.10) defined by the controllers is not sufficient for the stability of the closed-loop system (6.4) when $N > 2$ and $n_c > 1$.

Example 6.1 (Counterexample for three subsystems) *Consider the switched plant* (6.2) *with* $\mathcal{A} = \{A_1, A_2, A_3\}$*, where*

$$A_1 = \begin{pmatrix} 0 & 1 \\ -11.84 & -2.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -34.28 & -11.6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ -29.7 & -11 \end{pmatrix}$$

and $B_i = \begin{pmatrix} 0 & 1 \end{pmatrix}^\mathsf{T}$, $C_i = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $D_i = 0$ *for* $i = 1, 2, 3$. *And let the requested individual closed-loop poles be given by*

$$\Lambda_t = \big\{ -1 + 3i,\ -1 - 3i,\ -1.8,\ -8 \big\}.$$

It can be verified that the pole-placement requirement is satisfied by the following set of controllers

$$K_1 = \begin{pmatrix} 0 & 1 \\ -9.6 & -9.4 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 1 \\ -7.4 & -0.2 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 1 \\ -5.5 & -0.8 \end{pmatrix}$$

and $M_1 = \begin{pmatrix} 30.34 & -7.536 \end{pmatrix}$, $M_2 = \begin{pmatrix} -109.7 & 34.1 \end{pmatrix}$, $M_3 = \begin{pmatrix} -19.35 & 42.54 \end{pmatrix}$, *and* $L_i = \begin{pmatrix} 0 & 1 \end{pmatrix}^\mathsf{T}$, $J_i = 0$ *for* $i = 1, 2, 3$.

It can be numerically verified that $V(x) = x^\mathsf{T} P x$ *with*

$$P = \begin{pmatrix} 3.0745 & 0.0671 \\ 0.0671 & 0.4356 \end{pmatrix}$$

is a common quadratic Lyapunov function for the LTI systems $\Sigma_{K_1}, \Sigma_{K_2}, \Sigma_{K_3}$. *Hence, the switched system* (6.10) *consisting of the controllers is asymptotically stable for arbitrary switching.*

However we can find a switching signal for which the closed-loop switched system (6.4) *is unstable. The spectral radius*

$$\varrho\left(e^{0.22 H_3} e^{0.32 H_2} e^{0.72 H_1}\right) = 1.024$$

where H_i are the closed-loop system matrices (6.4b)*. Hence, the closed-loop system is unstable for the periodic switching signal of period $T = 1.26$, where one period is defined by the switching sequence* $(0,1)$, $(0.72,2)$, $(1.04,3)$*. Figure 6.4 shows a sample-output of the unforced switched system* (6.4) *with initial state* $x = \begin{pmatrix}0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\end{pmatrix}^{\mathsf{T}}$.

□

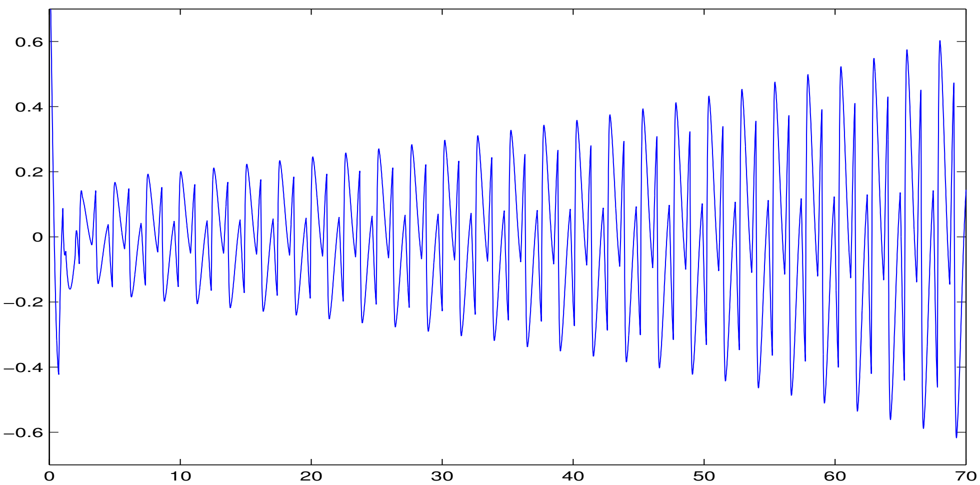


Figure 6.4: Sample-output of the system in Example 6.1 for the unstable switching sequence.

As shown by Example 6.1 asymptotic stability of the switched system (6.4) with $N > 2$ is not equivalent to the asymptotic stability of the switched system (6.10) formed by the controllers. However, it is still possible to reduce the dimension of the stability problem.

In the spirit of the previous section we consider a basis V_t of the common subspace of all the matrices H_i. Define the transformation matrix

$$T_1 = \begin{pmatrix} V_t & e_{n_p+n_c+1} & \dots & e_n \end{pmatrix}.$$

Applying the similarity transformation T_1 to the system matrices of (6.4) gives

$$\begin{aligned}
T_1^{-1} H_1 T_1 &= \operatorname{diag}(D_t, K_2, \dots, K_N) \\
T_1^{-1} H_2 T_1 &= \operatorname{diag}(D_t, K_2, \dots, K_N) + T_1^{-1}(H_2 - H_1) T_1 \\
&\vdots \\
T_1^{-1} H_N T_1 &= \operatorname{diag}(D_t, K_2, \dots, K_N) + T_1^{-1}(H_N - H_1) T_1
\end{aligned}$$

where $\sigma\big(D_t\big) = \Lambda_t$. Note that $(H_i - H_1)$ only has non-zero entries in the n_p-th row. Thus, $H_i - H_1 = e_{n_p} \tilde{h}_i$ where $\tilde{h}_i = h_{i n_p} - h_{1 n_p}$ denotes the differences between the n_p'th rows of H_i and H_1.

As implied by our construction, the differences between the matrices $T_1^{-1}H_iT_1$ are all multiples of the same columns. Furthermore, inspection of the n_p'th rows of the matrices H_i shows that $\tilde{h}_i$ only has nonzero entries in its first n_p+n_c positions and in the positions $n_p+(i-1)n_c+1,\dots,n_p+i\,n_c$. Hence, only the blocks of controller K_i are perturbed. So that for $i=2,\dots,N$ the matrices after similarity transformation are of the form

$$T_1^{-1}H_iT_1 \quad = \quad \begin{pmatrix} D_t & 0 & \dots & U_{1i} & 0 \\ 0 & K_2 & 0 & U_{2i} & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & K_i+U_{ii} & \\ & & & & 0 \\ & & & U_{Ni} & K_N \end{pmatrix}.$$

Where in particular the perturbation term $\begin{pmatrix} U_{1i}^\mathsf{T} & U_{2i}^\mathsf{T} & \dots & U_{Ni}^\mathsf{T} \end{pmatrix}^\mathsf{T}$ has rank one. We denote

$$\bar{K}_1 := \operatorname{diag}(K_2,\dots,K_N)\,,$$

and for $i=2,\dots,N$ the lower right $(N-1)n_c \times (N-1)n_c$-block of $T_1^{-1}H_iT_1$ by

$$\bar{K}_i := \begin{pmatrix} K_2 & 0 & U_{2i} & & 0 \\ & \ddots & \vdots & & \vdots \\ & & K_i+U_{ii} & & \\ & & \dots & \ddots & 0 \\ 0 & & U_{Ni} & & K_N \end{pmatrix}.$$

We then have the following result.

Theorem 6.6 *The following statements are equivalent:*

(i) The switched system given by the set of matrices $\{H_1,H_2,\dots,H_N\}$ is asymptotically stable.

(ii) The switched system given by the set of matrices $\{\bar{K}_1,\bar{K}_2,\dots,\bar{K}_N\}$ is asymptotically stable.

Proof. This follows from the preceding discussion. □

The above theorem reduces the order of the system for which stability needs to be established by $n_p + n_c$. Although, we cannot achieve a similarly large reduction as for the case of two subsystems, Theorem 6.6 can serve as a starting-point for further analysis.

6.2.3 Controllers with integrator

The stability results derived in the previous sections require that the constituent closed-loop systems Σ_{H_i} are stable LTI systems. Since the eigenvalues of the non-active controllers are part of the spectrum of H_i, we cannot apply the stability results when the controllers have integrators. In this section we show that this problem can be partially resolved by choosing a variation of the local-state controller-architecture.

For this step we shall assume that the local controllers have the same number of integrators. Then we can choose a controller-architecture where these integrators are shared by the controllers and therefore are always active in the closed-loop. In this section we shall investigate how the position of such joint integrator affects the structure of the closed-loop system matrices and show that the above stability results are applicable for a certain controller-architecture.

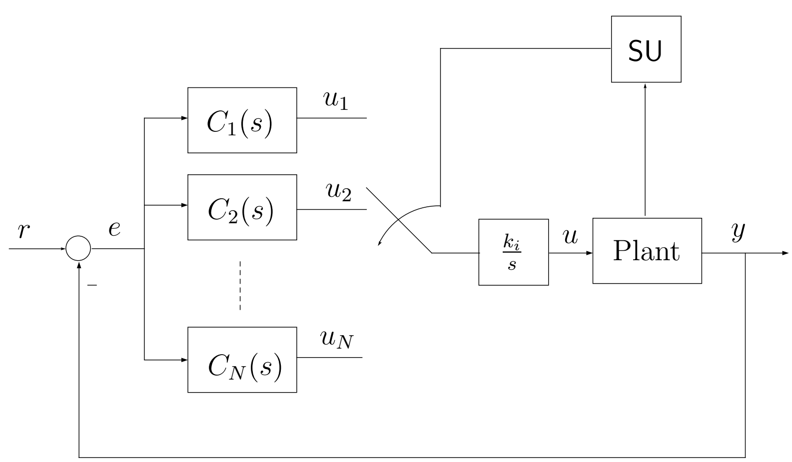


Figure 6.5: Local-state controller with joint integrator after the switch.

We begin with the case where the integrator is positioned after the switch of the controller bank. The closed-loop structure of that approach is depicted Figure 6.5, where k_i denotes the (possibly time-varying) integrator gain. The dynamics of the additional state u are given by $\dot{u} = k_i u_i$ where

$$u_i = M_i x_i - J_i C_i x_p + J_i r.$$

Defining the state-vector of the closed-loop system $x = \begin{pmatrix} x_p^\mathsf{T} & u^\mathsf{T} & x_i^\mathsf{T} & \cdots & x_N^\mathsf{T} \end{pmatrix}^\mathsf{T}$ yields the system matrices

$$H_i = \begin{pmatrix} A_i & B_i & 0 & \cdots & 0 \\ -k_i J_i C_i & 0 & M_i\delta_{i1} & \cdots & M_N\delta_{iN} \\ -L_1 C_1 & 0 & K_1 & 0 & 0 \\ \vdots & & \ddots & \ddots & \\ -L_N C_N & 0 & \cdots & 0 & K_N \end{pmatrix}, \qquad \forall\, i \in \mathcal{I}.$$

Note that the entries $M_i\delta_{ij}$ with the kronecker symbol are in the row of the integrator-state u and not as before in the same row as the system matrix of the plant. Thus the closed-loop system matrices differ by a rank of at least two, $rank\{H_i - H_j\} \geq 2$.

The analysis of the previous section relies on the existence of a set of common eigenvectors for the closed-loop system matrices in $\mathcal{H}$ (Lemma 6.3). However, the proof of the existence of such common eigenvectors depends on the fact that the matrices H_i differ pairwise by rank one. Therefore the results of the previous section are not applicable to the controller-structure with joint integrator after the switch.

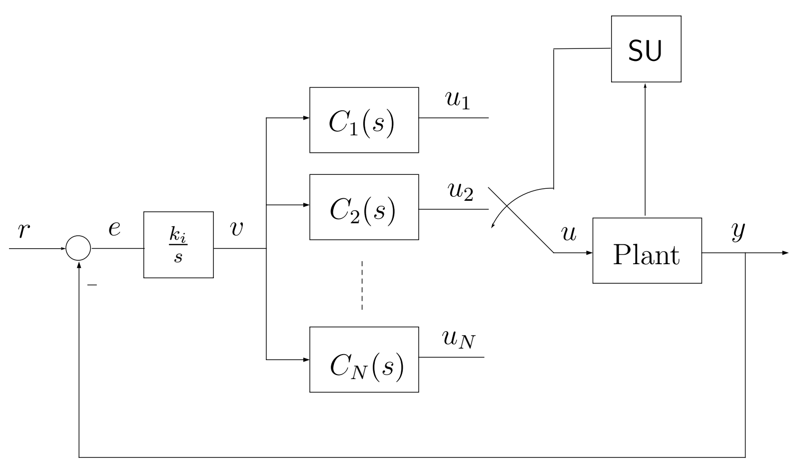


Figure 6.6: Local-state controller with joint integrator before the controller bank.

In some cases, this problem can be resolved by placing the joint integrator in front of the controller bank as shown in Figure 6.6. Define, the integrator state v where

$\dot{v} = k_i(-C_i x_p + r)$. The controller dynamics are then given by

$$\begin{aligned} \dot{x}_i &= K_i x_i + L_i v \\ u_i &= M_i x_i + J_i v \end{aligned}$$

Choosing the state-vector of the closed-loop system as $x = \begin{pmatrix} x_p^\mathsf{T} & v^\mathsf{T} & x_i^\mathsf{T} & \cdots & x_N^\mathsf{T} \end{pmatrix}^\mathsf{T}$ yields the system matrices

$$H_i = \begin{pmatrix} A_i & B_i J_i & B_i M_i \delta_{i1} & \cdots & B_N M_N \delta_{iN} \\ -k_i C_i & 0 & 0 & \cdots & 0 \\ 0 & L_1 & K_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \\ 0 & L_N & & 0 & K_N \end{pmatrix}, \qquad \forall\, i \in \mathcal{I}.$$

Consider the case where the output-matrices of the plant are constant up to some factor ν_i, i.e. $C_i = \nu_i C$ for all $i \in \mathcal{I}$. When choosing the integrator gain as $k_i = \frac{1}{\nu_i}$, the system matrices H_i differ by rank one.

It follows from the previous discussion that the stability results of this section are applicable, when the output-vectors of the process realisation satisfies $C_i = \nu_i C$. In this case we can choose a joint integrator for all local controllers positioned before the controller bank. When positioning the joint integrator after the switch we obtain closed-loop system matrices H_i for which $rank\{H_i - H_j\} = 2$ and thus the obtained stability results are not applicable.

6.3 Transient responses at steady-state switching

The dynamics of a switched system are given by the dynamics of the constituent subsystems and the switching action which orchestrates between them. While the dynamics of the constituent LTI systems are well understood, the influence of the switching action on the dynamical behaviour is difficult to specify.

In this section we are concerned with developing a better understanding of the influence of the switching action on the dynamic behaviour of the switched system. Therefore we eliminate the dynamics of the subsystems by considering the switching system in steady-state where the dynamics of the subsystems have converged.

Definition 6.7 *The state-vector $\hat{x}$ of the LTI system with stable transfer function*

$G(s) = c^{\mathsf{T}}\big(sI - A\big)^{-1}b + d$ *and a given constant reference input* $r(t)$ *is called steady-state if* $\dot{\hat{x}} = 0$.

The steady-state of the LTI system with realisation $\big(A, b, c^{\mathsf{T}}, d\big)$ for the constant reference-input r is given by

$$\begin{aligned} 0 &= A\hat{x} + br \\ \hat{x} &= -A^{-1}br\,. \end{aligned}$$

The difference of the output $y = c^{\mathsf{T}}\hat{x} + dr$ and the constant reference-input r is called static steady-state error. Note that the dynamics of the LTI system converge asymptotically to the equilibrium and therefore steady-state is only attained for $t \to \infty$. However in practice it is common to neglect the dynamics of the LTI system after some finite settling time.

In this section we consider the input-output description of the switched closed-loop system

$$\dot{x}(t) = H_{\sigma(t)}x(t) + \bar{B}_{\sigma(t)}r(t) \qquad (6.11a)$$

$$y(t) = \bar{C}_{\sigma(t)}x(t) \qquad (6.11b)$$

where $\sigma : \mathbb{R}^+ \to \mathcal{I}$, $H_i \in \mathcal{H}$, $\bar{B}_i \in \mathcal{B}$ and $\bar{C}_i \in \mathcal{C}$. The matrices $H_i, \bar{B}_i, \bar{C}_i$ depend on the chosen controller architecture and on the realisation of the transfer functions $P_i(s)$ and $C_i(s)$. Recall that we can only choose the realisations of the controller transfer functions $C_i(s)$ independently of each other. We shall assume that the controllers are designed such that the static steady-state error is zero for all modes.

In this section we analyse properties of the transient behaviour of the switched system that is induced solely by the switching action. In particular we shall consider transient responses that occur due to switching when the subsystems have reached steady-state. This effect is illustrated by the following example.

Example 6.2 *Consider the plant* (6.2) *with switched linear dynamics and two modes. The dynamics of the constituent modes are described by the transfer functions*

$$P_1(s) = \frac{1}{s+14}\,, \quad P_2(s) = \frac{1}{s+3}\,.$$

We choose first-order controllers for the control of this process. The closed-loop poles are given by $\Lambda_t = \big\{-2, -20\big\}$. *The controllers with transfer functions*

$$C_1(s) = \frac{-72}{s+8}\,, \quad C_2(s) = \frac{s-14}{s+18}$$

satisfy this pole assignment. For compensation of the static steady-state error we use the pre-filter gains $F_1 = -0.5556$ *and* $F_2 = -2.3529$.

The resulting closed-loop system dynamics for each mode is identical and given by the transfer function

$$T_i(s) \quad = \quad \frac{C_i(s)P_i(s)}{1 + C_i(s)P_i(s)} \quad = \quad \frac{40}{s^2 + 22s + 40} \, .$$

According to our assumptions, the state-space description in each plant-mode is given in control canonical form. We apply the controller in local-state architecture, choosing also the local controllers in control canonical form. For the closed-loop system (6.11) *we obtain*

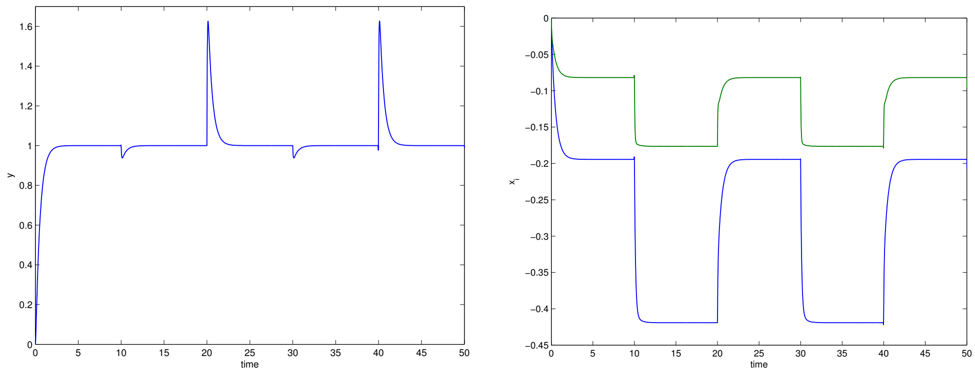
$$H_1 = \begin{pmatrix} -14 & -72 & 0 \\ -1 & -8 & 0 \\ -1 & 0 & -9 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} -3 & 0 & -17 \\ -1 & -8 & 0 \\ -1 & 0 & -9 \end{pmatrix}.$$

The input matrices are given by $\bar{B}_1 = F_1\begin{pmatrix}0 & 1 & 1\end{pmatrix}^{\mathsf{T}}$, $\bar{B}_2 = F_2\begin{pmatrix}0 & 1 & 1\end{pmatrix}^{\mathsf{T}}$ *the output matrices are* $\bar{C}_i = \begin{pmatrix}1 & 0 & 0\end{pmatrix}$, $i = 1, 2$. *Note, that the autonomous switched closed-loop system is asymptotically stable for arbitrary switching signals (Corollary 6.4) and thus the switched input-output system is bounded-input bounded-output stable [Rug96].*

Figure 6.7a shows a step response of the closed-loop system. Here a switching signal $\sigma(t)$ *is chosen that changes the mode every 10 time-units. At every switching instant we can observe a significant transient response before the output reaches its reference value again.*

Note that the switching is slow enough to allow the system to reach practically steady-state between two consecutive switching instances, i.e. the observed transient-peaks are not due to unsettled control signals or states. $\square$

The closed-loop transfer functions and therefore the input-output behaviour in both modes of the switched system are identical. Hence, the transients are clearly a result of the different eigenvectors of the system matrices H_1 and H_2. Therefore we can expect different transient behaviour depending on the realisations of the controller transfer functions. The objective in this section is to find conditions for which transient-free switching at steady-state can be achieved.



(a) (b)

Figure 6.7: Step-response of the closed-loop system in Example 6.2 with the switching instances $10, 20, 30, \ldots$. Part (a) shows the evolution of the output $y(t)$, Part (b) shows the two controller-states $x_1(t)$ and $x_2(t)$.

We begin our discussion by investigating the cause of the transients that we observe in the above example. Consider the switched system with local-state controller. Figure 6.7b shows the evolution of the controller-state x_1. For the first ten time-units the plant is in mode 1 and thus controller $C_1(s)$ is active. After roughly 1.5 time-units the system has settled. But when the plant switches into mode 2 and controller $C_2(s)$ is active in the closed loop, the state of the non-active controller $C_1(s)$ settles on a different value. When the plant switches back into mode 1, the controller has the 'wrong' state and we observe a transient response until the controller attains its steady-state for that mode. Thus, the switched system has different steady-states for each mode. In view of that one could argue that the *switched* closed-loop system actually never reaches a steady-state since the controller-states alternate between an 'active' steady-state and a 'passive' steady-state. We shall therefore define the common steady-state for a set of LTI systems.

Definition 6.8 *The stable LTI systems with realisations* $\big(A_i, B_i, C_i, D_i\big)$ *with* $i \in \mathcal{I}$ *have the common steady-state* $\hat{x}$ *for the constant input* r *if*

$$\hat{x} \quad = \quad -A_i^{-1} B_i r \quad = -A_j^{-1} B_j r \qquad \forall\, i, j \in \mathcal{I}\,. \tag{6.12}$$

Note, that the systems $\big(A_i, B_i, C_i, D_i\big)$ either have common steady-states for all constant inputs $r \in \mathbb{R}$ or have no common steady-state.

With these observations it is not hard to show that the switched system has no transients due to switching at steady-state if the constituent systems have a common steady-state.

Theorem 6.9 *The switched input-output system* (6.11) *has no transient responses when switching at steady-state, if and only if the constituent systems* $\left(H_i, \bar{B}_i, \bar{C}_i\right)$ *have a common steady-state for all constant inputs* $r \in \mathbb{R}$.

Proof. Let t_{i_k} be the switching instant in which the system switches from mode i to mode j and let $\lim_{t \to t_{i_k}} x(t) = \hat{x}^{(i)} = -H_i^{-1}\bar{B}_i r$ where $\hat{x}^{(i)}$ denotes the steady-state of the LTI system in mode i for the constant input r. During the interval $t_{i_k} \leq t < t_{i_{k+1}}$ the dynamics of the switched system are given by $\left(H_j, \bar{B}_j, \bar{C}_j\right)$. Hence,

$$\begin{aligned} \dot{x}(t_{i_k}) &= H_j \hat{x}^{(i)} + \bar{B}_j r \\ &= -H_j H_i^{-1} \bar{B}_i r + \bar{B}_j r \end{aligned}$$

The switched system shows no transient response if and only if $\dot{x}(t_{i_k}) = 0$. Thus

$$\begin{aligned} 0 &= -H_j H_i^{-1} \bar{B}_i r + \bar{B}_j r \\ H_i^{-1} \bar{B}_i r &= H_j^{-1} \bar{B}_j r \end{aligned}$$

The latter equality is the condition for the exists of a common steady-state for the systems $\left(H_i, \bar{B}_i, \bar{C}_i\right)$ and $\left(H_j, \bar{B}_j, \bar{C}_j\right)$. □

A trivial requirement for transient-free switching at steady state is that the controllers ensure that the closed-loop system has the same static steady-state error $e_0 = \lim_{t\to\infty} r - y(t)$ for all mode $i \in \mathcal{I}$. Without loss of generality we assume that $e_0 = 0$. The output at steady-state is given by $y = C_i \hat{x}_p^{(i)}$. Thus, a necessary condition for the existence of a common steady-state for all plant-modes is that the first entry of the output-vector C_i is constant for all i.

6.3.1 Transient-free control-design for local-state controllers

In this section we derive conditions that ensure that (6.12) is satisfied, and can be incorporated into the pole-placement design. As before we assume that the dynamics of the plant in each mode are given in control-canonical form. Thus the system matrices A_i are in companion form and only the n_p-th entry of B_i is non-zero. It follows that only the first entry of the steady-state of each plant mode is non-zero.

In this section we focus on the control of the process with one scalar input u and one scalar output y. The dynamics are given by linear time-varying differential equation

$$y^{(n)} = \sum_{k=0}^{n-1} q_k(t) y^{(k)} + p_0(t) u \qquad (6.13)$$

where $p_0(t), q_k(t)$ are piecewise constant functions that can take on values in the finite sets $p_0(t) \in \{p_{01}, \ldots, p_{0N}\}$, and $q_k(t) \in \{q_{k1}, \ldots, q_{kN}\} \;\forall\; k = 0, \ldots, n-1$. The discontinuities occur at the same time-instances such that $p_0(t) = p_{0\sigma(t)}$ and $q_k(t) = q_{k\sigma(t)}$ with $\sigma : \mathbb{R} \to \mathcal{I} = \{1, \ldots, N\}$. Hence, in any given point in time, the dynamics of the plant can be described by the transfer function

$$Y(s) \;=\; P_i(s)U(s) \;=\; \frac{p_{0i}}{q_i(s)}U(s), \qquad i = \sigma(t) \in \mathcal{I} \tag{6.14}$$

where $q_i(s)$ is a polynomial of the complex variable $s \in \mathbb{C}$ with corresponding coefficients q_{ki}. Note that the transfer functions $P_i(s)$ have no zeros.

Defining the state-vector $x_p = \begin{pmatrix} y & \dot{y} & \ldots & y^{(n)} \end{pmatrix}^\mathsf{T}$, we obtain the state-space representation of the switched plant

$$\begin{aligned} \dot{x}_p(t) &= A_{\sigma(t)}x_p(t) + B_{\sigma(t)}u(t) \\ y(t) &= Cx_p(t) \end{aligned}$$

where $C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$ and

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -q_{0i} & -q_{1i} & \cdots & \cdots & -q_{n-1i} \end{pmatrix}, \qquad B_i = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ p_{0i} \end{pmatrix}.$$

The objective is to design controllers $C_i(s)$ for each mode such that the closed-loop system has no transient responses at the switching instances when switching at steady-state. For the control-design we choose the transfer function description of the controllers such that the obtained conditions for transient-free switching can be combined with the pole-placement approach.

Theorem 6.9 requires for transient-free switching that the constituent closed-loop systems $(H_i, \bar{B}_i, \bar{C})$ have a common steady-state. We reformulate this condition by considering the states of each individual controller separately. Since we assume that the individual controllers ensure that the static steady-state error is zero for each mode, it follows that the steady-state $\hat{x}_p$ of the plant is common to each subsystem $(H_i, \bar{B}_i, \bar{C})$.

Consider now the steady-state $\hat{x}_j$ of controller $C_j(s)$. Since we realise the transfer function in control canonical form, only the first entry of $\hat{x}_j$ is non-zero. The controller-output is given by $u_j = M_j x_j$, whether the controller is active in the closed-loop or not. It follows that the steady-state of controller $C_j(s)$ is common to all subsystems $\left(H_i, \bar{B}_i, \bar{C}\right)$ $i \in \mathcal{I}$ if and only if the steady-state controller-output $\hat{u}_j = \lim_{t\to\infty} u_j(t)$ is independent of the active controller in the loop.

At any time-instant, the controller-output $U_j(s)$ is given by

$$U_j(s) = \frac{C_j(s)}{1 + C_i(s)P_i(s)} F_i \, R(s)$$

where i is the current mode of the plant. Define g_{ji} as the steady-state gain from the reference input r to the controller-output u_j when the plant is in mode i,

$$g_{ji} = \lim_{s\to 0} \frac{C_j(s)}{1 + C_i(s)P_i(s)} F_i \, .$$

It follows that the controller $C_j(s)$ attains the same steady-state in each mode $i \in \mathcal{I}$ if and only if $g_{ji} = g_j \in \mathbb{R}$ for all $i \in \mathcal{I}$.

Corollary 6.10 (Transient-free switching for local-state controllers) *Given the process* (6.13) *with N modes and N stable controllers such that the closed-loop LTI systems $\left(H_i, \bar{B}_i, \bar{C}\right)$ are stable and have the static steady-state error $e_0 = 0$ for all modes $i \in \mathcal{I}$. Let the switched system* (6.11) *describe the closed-loop dynamics with local-state controller structure and the controllers be realised in control canonical form. Then the switched closed-loop system* (6.11) *has no transient response when switching at steady-state if and only if $g_{ji} = g_j \in \mathbb{R}$ for all $i, j \in \mathcal{I}$.*

Integrators of the open-loop dynamics play an important role for the steady-state behaviour of LTI feedback systems. We shall now consider three common cases and investigate whether transient-free switching at steady-state is possible. In order to simplify the classifications, we parameterise the transfer functions of the controllers and plants as follows

$$C(s) = \frac{N_C(s)}{D_C(s)} K_C s^{l_c} \tag{6.15}$$

$$P(s) = \frac{N_P(s)}{D_P(s)} K_P s^{l_p} \tag{6.16}$$

where $\frac{N_C(0)}{D_C(0)} = \frac{N_P(0)}{D_P(0)} = 1$ and $l_c, l_p \in \mathbb{Z}$.

Case 1 Consider the case where the plant has a pure integrator in all modes, i.e. $l_p < 0$, and the controllers C_i have no integrator, i.e. $l_c = 0$. The steady-state gain

g_{ji} is then given by

$$g_{ji} = \lim_{s\to 0} \frac{K_{Cj}}{1 + K_{Ci}K_{Pi}s^{l_p}} F_i \,.$$

When the open-loop transfer function $C_i(s)P_i(s)$ has an integrator, the pre-filter gain $F_i = 1$. It follows that the steady-state gain $g_{ji} = 0$ for all $i, j \in \mathcal{I}$. Hence, for any set of controllers that result in stable closed loop systems we obtain transient-free switching at steady-state.

Case 2 Secondly, consider the case where neither the plant nor the controllers have an integrator ($l_c = l_p = 0$). Since the open-loop has no integrator a pre-filter for each mode is required to ensure that the static steady-state error is zero. The pre-filters are given by $F_i = \frac{1+K_{Ci}K_{Pi}}{K_{Ci}K_{Pi}}$. For the steady-state gains g_{ji} we obtain

$$g_{ji} = \frac{K_{Cj}}{1 + K_{Ci}K_{Pi}} F_i \,.$$

Substituting F_i yields after cancellation

$$g_{ji} = \frac{K_{Cj}}{K_{Ci}K_{Pi}} \,.$$

In this case transient-free switching at steady-state is only guaranteed if the open-loop steady-state gains satisfy

$$K_{Ci}K_{Pi} = K \qquad \forall\, i \in \mathcal{I} \tag{6.17}$$

for some chosen $K \in \mathbb{R}$.

Case 3 Consider the case where the plant has no integrator ($l_p = 0$) and the controllers have an integrator ($l_c < 0$). To ensure that the controllers do not wind up when non-active we have to use one of the joint-integrator structures as described in Sections 6.2.3. Consider first the case where the joint integrator is positioned after the switch, right before the plant. In terms of our steady-state analysis this case is equivalent to the first case, where the plant has an integrator as in case 1.[2] Hence, there are no further restrictions on the controllers to ensure transient-free switching at steady-state.

For the implementation where the joint integrator is in front of the controller bank, we obtain

[2]Note that this simplification is only valid because we consider static behaviour of the system. For the stability analysis such conclusion requires further justification.

$$\begin{aligned} g_{ji} &= \lim_{s\to 0} \frac{K_{Cj}s^{l_c}}{1+K_{Ci}s^{l_c}K_{Pi}} \\ &= \frac{K_{Cj}}{K_{Ci}K_{Pi}} . \end{aligned}$$

Hence, this case is equivalent to the second case. Transient-free switching is achieved when $K_{Ci}K_{Pi} = K \in \mathbb{R}$ for all $i \in \mathcal{I}$.

We summarise that in all considered cases we can design controllers for which the closed-loop switched system has no transient responses when switching at steady-state. There is no further conditions on the control-design when the plant has an integrator or the joint controller-integrator is positioned after the switch. In the other two cases we require that $K_{Ci}K_{Pi} = K_{Cj}K_{Pj}$ for all $i, j \in \mathcal{I}$.

Note that condition (6.17) only concerns a single parameter of each individual controller. Therefore, we can always satisfy the pole-placement requirement and condition (6.17) by simply choosing a controller with on additional degree of freedom. Thus whenever the pole-placement is feasible, we can design controllers that result in transient-free switching at steady-state. We shall demonstrate this by the following example.

Example 6.3 *With condition* (6.17) *we can design a controller for the plant in Example 6.2 that guarantees transient-free switching. To satisfy the pole-placement requirement simultaneously, we need to employ controllers with one more degree of freedom, we choose* $C_i(s) = \frac{a_{1i}s+a_{0i}}{s+b_{0i}}$. *With the same closed-loop poles required* ($\Lambda_t = \{-2, -20\}$) *and choosing the constant* $K = 2$ *in* (6.17) *we obtain the following controllers*

$$C_1(s) = \frac{2.286s - 40}{s + 5.714}, \quad C_2(s) = \frac{-7.667s - 40}{s + 26.67}.$$

with the pre-filter gains $F_1 = F_2 = -1$.

$$H_1 = \begin{pmatrix} -16.29 & -53.06 & 0 \\ -1.0 & -5.71 & 0 \\ -1.0 & 0 & -26.67 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 4.67 & 0 & 164.44 \\ -1.00 & -5.71 & 0 \\ -1.00 & 0 & -26.67 \end{pmatrix}.$$

The input matrices are given by $\bar{B}_1 = \begin{pmatrix} -2.2857 & -1 & -1 \end{pmatrix}^\mathsf{T}, \bar{B}_2 = \begin{pmatrix} 7.6667 & -1 & -1 \end{pmatrix}^\mathsf{T}$ *the output matrices are* $\bar{C}_i = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $i = 1, 2$.

Figure 6.8 shows the step-response of the switched system when switching every 10 time-units. The switching of the plant mode is not visible at the controller output.

This is due to the fact that both controller states reach their steady-state before the first mode-switch occurs at 10 time-units (solid lines in Figure 6.9). Since, the steady-state of each controller is the same regardless whether the controller is active or passive in the loop, the controller outputs remain constant. Only the plant-input (dashed line) is switched to the correct value. □

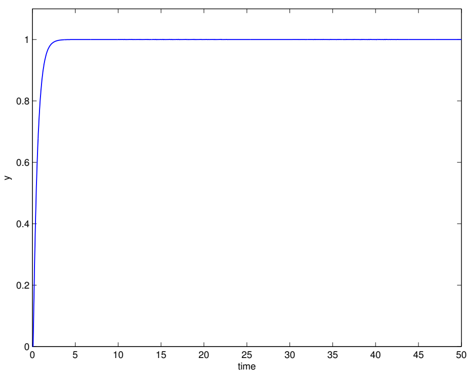


Figure 6.8: Step response of the control system with transient-free local-state controller (switching instances $10, 20, 30, \ldots$)

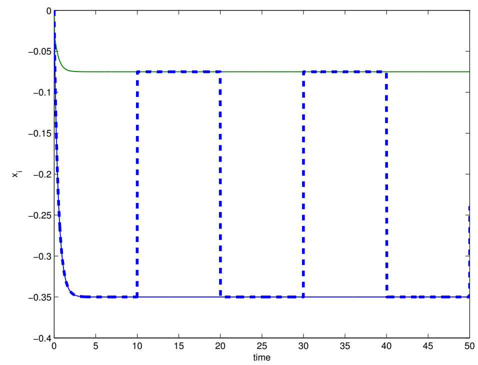


Figure 6.9: Individual controller states (solid lines) and plant-input (dashed line).

6.4 Relation to alternative approaches

For the analysis in this chapter we have chosen a particular controller-architecture, where the controller is implemented as a set of LTI systems with individual states. Of course, there are many other ways of implementing the constituent controllers. In this section we shall briefly relate two alternative approaches for the implementation of the switched controller to the results of this chapter.

Global-state architecture

A common approach in adaptive control, for example, is to realise the set of controllers as a switched system on its own [Mor98, ABB$^+$00, HLM$^+$01]. The controller dynamics are then given by

$$\dot{x}_c(t) = K_{\sigma(t)} x_c(t) + L_{\sigma(t)} e(t) \tag{6.18a}$$

$$u(t) = M_{\sigma(t)} x_c(t) + J_{\sigma(t)} e(t) \tag{6.18b}$$

where $K_{\sigma(t)} \in \{K_1, \ldots, K_N\}$, $L_{\sigma(t)} \in \{L_1, \ldots, L_N\}$, $M_{\sigma(t)} \in \{M_1, \ldots, M_N\}$, and $J_{\sigma(t)} \in \{J_1, \ldots, J_N\}$. Note that the controller parameters simultaneously switch at the switching instances. In contrast to the local-state controller the dynamics act on the same state $x_c \in \mathbb{R}^{n_c \times n_c}$. Therefore we shall refer to this controller structure as *global-state* controller.[3]

Applying the global-state controller to the switched plant (6.2) results in switched closed-loop systems with state-vector $x = \begin{pmatrix} x_p^\top & x_c^\top \end{pmatrix}^\top \in \mathbb{R}^{n_p+n_c}$ and system matrices

$$H_i = \begin{pmatrix} A_i - B_i J_i C_i & B_i M_i \\ -L_i C_i & K_i \end{pmatrix}. \tag{6.19}$$

The results obtained in this chapter for local-state controllers are not applicable to this controller architecture. Since all controllers share the same state, the separation of the subspaces used in the derivations is not possible.

A direct comparison of the stability properties of the systems with the two controller architectures is not straight forward, since the closed-loop stability of the global-state controller is certainly dependent on the realisation chosen for the individual controllers. However, it should be noted that the stability of the resulting closed loop system can be different for the two controller architectures, when the same controller-realisations are applied [LLMS02]. For example it is not hard to find examples with control canonical realisation of the individual controllers, for which the local-state controller architecture results in a stable closed-loop system whereas the system with the global-state controller is unstable.

We shall still note some important differences between the two controller architectures. Assuming minimal controller realisations the most obvious difference of the resulting closed-loop systems is the system order. While the local-state controller architecture results in closed-loop systems of order $n_p + N n_c$, the system with global-state controller has order $n_c + n_p$. Thus, using the global-state controller results in closed-loop systems whose order is independent of the number of subsystems. However, such assessment is only valid when the realisations of the individual controllers have the same order for the global-state and the local-state architecture. This is not necessarily given, since different controller realisations might be preferable for each controller architecture.

Moreover, a greater system-order does not necessarily imply that the stability analysis is more complex. Equally important is the rank of the differences of the system

[3]In the multiple adaptive control literature this controller architecture is also known as state-shared multicontroller.

matrices $H_i - H_j$, $i \neq j$. For the local-state controller architecture we obtain $rank\{H_i - H_j\} = 1$ whereas the global-state architecture yields $rank\{H_i - H_j\} \geq 2$, depending on the controller-realisations. Loosely speaking, this means that the system matrices obtained using the local-state controller are structurally more similar to each other. That means that the differences of the vector fields are in the same one-dimensional space. For the case of the global-state implementation, the difference of the vector fields belong to a higher dimensional space and are therefore less structured.

State-reset approach

An interesting extension of these controller architectures has been recently suggested in [HM02]. Here the authors consider a global-state controller where the controller state is reset to a desired value whenever its dynamics change. This could be interpreted as a mixture of the global- and the local-state architecture. While the individual controllers share the same state space, their dynamics are to a certain degree decoupled since at any switching instant the initial state can be chosen freely. Clearly, such approach allows more control over the transient behaviour of the switched system. This problem has been considered in [PV03]. The results in both publications consider the control of a time-invariant plant and are therefore not applicable for the case considered in this chapter. However, the approach appears to be very effective and suitable to be adopted for our problem.

6.5 Conclusions

In this chapter we considered a typical control problem for a class of switched single-input single-output systems with N subsystems. For the control-design we focussed on two important objectives: firstly, the stability of the closed-loop system for arbitrary switching, and secondly the transient behaviour of the closed-loop system at the switching instances.

For the control of the process we have chosen a switched controller with local-states. This controller architecture has the conceptional advantage that the resulting close-loop system matrices have differences $H_i - H_j$ of rank one for all $i, j \in \mathcal{I}, i \neq j$, which simplifies the stability analysis of the switched system. In conjunction with the assumption that the poles of the closed-loop transfer function are constant for all modes, we derived conditions for the stability of the switched system. For the case that the pole-placement requirement is feasible for first-order controllers and the target-poles are real we have shown that the resulting closed loop system is always stable given that the first-order controllers have poles in the open left half-plane. For systems with controllers of arbitrary order and two subsystems, the stability analysis reduces to the analysis of the autonomous switched system whose constituent system matrices are the system matrices K_1, K_2 of the controllers. This is a significant simplification. For example, in the context of this chapter we apply the matrix-pencil results of Chapter 3 to design switched control systems of order up to five.

For systems with more than two subsystems which require controllers of order greater than one, it was shown that such simplification is in general not possible. In this case the asymptotic stability of the n-th order closed-loop system reduces to the asymptotic stability of order $n-(n_p+n_c)$. However, the system matrices of the reduced switched system have a significant structure that might prove beneficial for further research.

In the second part of this chapter we analysed the transient behaviour of the closed-loop system that is induced by the switching action. We have derived conditions for the design of the individual controllers that guarantee transient-free switching when the system is in steady-state. These conditions can be incorporated into the design procedure of the controllers. Increasing the order of the controllers by one, we can achieve transient-free switching and satisfy the pole-placement requirement simultaneously. This allows to design controllers that achieve both, stability and transient-free switching at steady-state, using the design methods proposed in this chapter.

The stability analysis in this chapter depends fundamentally on the assumption that the poles of the closed-loop transfer function are invariant while switching. This requires that the respective controller outputs are instantaneously activated whenever the plant-mode changes. From a practical point of view this is an unrealistic assumption. In most applications there will be a certain time-delay between the mode-switch of the plant and the switching of the control signal. The impact of such delays on the stability of the closed-loop system are an important problem and are subject of future research.

An open question is also how the realisations of the transfer functions effect the results in this chapter. Throughout this chapter we assume that the individual controllers are realised in control canonical form. In particular the derivation of the result for two subsystems relies on this fact. While this is a realistic assumption a different choice of the realisation might provide better performance or stability properties. It is well known that the eigenvectors of the constituent system matrices play an important role for the stability of a switched linear system. Since we can choose the controller realisations independently of each other, it might be possible to find conditions on the realisations that simplify the stability analysis.

Chapter 7

Conclusions

In this final chapter we present some brief concluding remarks, summarise the work of earlier chapters, and highlight the main contributions made in the course of the thesis.

Switched linear systems exhibit complex dynamical behaviour which can be critical for their stability properties. In this thesis we analyse the stability of several classes of switched linear systems and contribute to a better understanding of their stability properties.

In Chapter 3 we introduced a novel type of piecewise linear Lyapunov function that is more suitable for the analysis of a certain class of switched system. In conjunction with quadratic Lyapunov functions we derive a compact stability condition that is readily applicable and provides some insight into the dynamical behaviour of the switched linear system. When there exists a common unic Lyapunov function, this Lyapunov function is directly available and can therefore be used as a guideline for the control design.

In Chapter 4 we established the relation of the asymptotic stability of switched systems with arbitrary switching signals and the robust control problem of the Lur'e system. We generalised a result in [SN03a] to derive spectral conditions of classical stability results for Lur'e systems. This result can be applied for a large number of conditions, some of which are briefly discussed in the chapter. In particular, we have demonstrated that the spectral formulation of the frequency-domain conditions can simplify the control design under constraint conditions.

Chapter 5 is dedicated to finding non-conservative stability conditions for switched SISO systems (and by equivalence, absolute stability of SISO Lur'e systems). The approach here is to identify the existence of a periodic solution and to infer asymptotic stability of the system from its absence. While the review of the literature on this problem reveals that this implication is true for systems of order up to three, its validity for higher order systems remains an open problem. However, it is shown that the largest range of parameter variation for which a given nominal system is stable can be approximated arbitrarily closely. Unfortunately, this requires the determination of the supremum of the real spectrum of an infinite set of operators which is infeasible in practice. An attempt to resolve this problem is to approximate the operators by truncated matrices. It is demonstrated that this method can yield good approximations for the existence of periodic solution for a given switching signal. The pertinent problem in this context is to identify a class of switching signals that is necessary and sufficient for the asymptotic stability of the system with arbitrary switching signals. For second- and third-order systems such a class is given by periodic signals with two discontinuities per period. This restriction makes the derived stability condition computational feasible. There are some indications that this class of signals is also sufficient for determining asymptotic stability of higher order systems . However, the generalisation of this result remains an open problem.

In the last chapter we consider a typical control task for switched processes and suggest a method for the design of a switched controller. We show that the stability analysis of the closed-loop system can be significantly simplified when using the proposed switched controller structure. Moreover, we derive conditions that ensure transient-free switching at steady-state. These conditions can be applied to several types of stable controllers for the system, i.e. whenever there is a set of controllers such that the closed-loop system is stable we can also achieve transient-free switching.

Notations and Abbreviations

Notations

$\mathbb{R}^+$	set of non-negative real numbers
$A^\mathsf{T}, c^\mathsf{T}$	transpose of the matrix A, transpose of the vector c
I	identity matrix of appropriate dimension
$\mathrm{Re}\{x\}$	real part of the complex number x
$\mathrm{Im}\{x\}$	imaginary part of the complex number x
$\mathrm{tr}A$	trace of the matrix A
$\mathrm{tr}^2 A$	square of the trace of the matrix A, $(\mathrm{tr}A)^2$
tr_i	$\mathrm{tr}A_i$ (only used for some calculations in Chapter 3)
Δ_i	$\det A_i$ (only used for some calculations in Chapter 3)
$Q < 0$, $(Q > 0)$	matrix Q is positive (negative) definite
$Q \leq 0$, $(Q \leq 0)$	matrix Q is positive (negative) semi-definite
$\sigma(A)$	spectrum of the matrix A
$\varrho(A)$	spectral radius of the matrix A, i.e. $\max \lvert\sigma(A)\rvert$
$\mathrm{diag}(A, B, C)$	block diagonal matrix with blocks given by the square matrices A, B, C
$\sigma_\alpha[A_1, A_2]$	matrix pencil $\alpha A_1 + (1-\alpha)A_2$, where $\alpha \in [0, 1]$
$\sigma_\gamma[A_1, A_2]$	matrix pencil $A_1 + \gamma A_2$, where $\gamma \geq 0$
Σ_A	linear time-invariant system $\dot{x}(t) = Ax(t)$
$\Sigma_\mathcal{A}$	switched linear system with the set of system matrices $\mathcal{A}$ and arbitrary switching signals
$\Sigma_{\mathcal{A},S}$	switched linear system with the set of system matrices $\mathcal{A}$ and the set of switching signals in $\mathcal{S}$
$co(\mathcal{A})$	convex hull of $\mathcal{A}$

Abbreviations

CLF	common Lyapunov function
CQLF	common quadratic Lyapunov function
CULF	common unic Lyapunov function
LTI	linear time-invariant
PLF	piecewise linear Lyapunov function
LMI	linear matrix inequality
SISO	single-input single-output
BIBO	bounded-input bounded-output

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